Time-Additive Consumption-Wealth Utility

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I Introduction

Time-additive utility of consumption functions are widely used in intertemporal models and empirical work in Economics and Finance. The additive structure is compatible with dynamic programming and allows for simplifications in the analysis. It is well-known, however, that a multi-attributed utility which is additively separable displays strong utility independence. This utility independence results in a number of properties for time-additive utility that have been criticized, particularly in the last quarter century. Among the concerns are: (i) Intertemporal risk aversion measuring how risks at different times interact is always zero. (ii) Consumption at one date does not affect the utility realized from consumption at other dates. (iii) There is complete neutrality over the timing of the resolution of risk. (iv) The intertemporal elasticity of substitution cannot be specified independently from risk aversion over gambles with immediate resolution. Each of these concerns has been addressed in the literature by a weakening of the expected utility axioms or suggestions for alternative representations of lifetime utility.[1]

The properties of multi-attributed utility functions have been a concern almost from the beginning of expected utility theory, but the problems associated with multiple goods has largely been ignored in intertemporal financial models by assuming a single consumption good or homothetic utility functions which allows for an aggregate composite good. Nevertheless, even with a single good, consumption at different dates must be treated separately but somehow combined into a single utility measure. Assuming a time-additive structure is the most common way to do this, but additively separable utility has no multi-attributed risk aversion since $\partial^2 U/\partial C_t\,\partial C_\tau = 0$ for any two times.[2] This cross partial derivative also directly determines the intertemporal elasticity of substitution and can be related to the preference for resolution timing.

The notion that past consumption may affect current tastes is very old, dating back at least to Marhsall (1920). Duesenberry (1949) is probably the first to examine the implications of habit persistence analytically. Constantinides (1990) and Ingersoll (1992) examined a particular functional form of cardinal utility for the effects of habit formation in a continuous-time consumption-portfolio problem. The analysis in this paper is close in spirit to the habit formation literature, particularly those last two papers, but it is the interaction with future rather than past consumption that is of concern here.

Concerns about the timing of the resolution of risk can be either inherent or induced. When discussing income streams with permitted borrowing and lending it is natural to favor earlier risk resolution as this can only help planning consumption. The inherent preference for earlier or later resolution of consumption uncertainty is different. Kreps and Porteus (1978, 1979) analyzed this problem and derived an ordinal aggregation of consumption and certainty equivalents that describes resolution preferences.

This same structure can handle the severance of the intertemporal elasticity of substitution and atemporal risk aversion. It has been particularly popularized by Epstein and Zin (1989,

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[1] Most of these ideas have been examined by many researchers and there is no attempt here to make an exhaustive list. The papers mentioned are largely those analyzing specific alternative representations, along the lines of this paper, rather than the initial or axiomatic developments.

[2] See Richard (1975) for the derivation of this measure of multi-attributed risk aversion.

1991) as recursive utility. Duffie and Epstein's (1992a, b) stochastic differential utility is the continuous-time version of recursive utility. It will be seen that the formulation proposed in this paper is closely related.

This paper introduces another utility form, time-additive felicity of consumption and wealth, which can be used to study such issues. It makes many predictions similar to those of recursive (or stochastic differential) utility but maintains the intuitive time-additive structure that is also amenable to dynamic programming.

Section II of this paper introduces time-additive consumption wealth felicity and discusses its merits. Section III demonstrates how consumption-wealth felicity can be used to derive the relation between the consumption and wealth stochastic processes and the interest rate and assets' risk premia. The predicted relations do not suffer from an equity premium paradox. Section IV analyzes the intertemporal consumption-portfolio problem and discusses the aggregation of individual demands into an equilibrium. Section V solves some simple portfolio problems for homogeneous utility. Section VI concludes.

II The Model Basics

In the time-additive expected utility model, lifetime utility is the sum (or integral) of the contribution to utility from each period. To avoid confusion of terms we follow the common practice and refer to this single-period contribution as felicity. In the simplest models, felicity depends on only consumption and is additive over time. We will adopt the same simple time-additive structure. Lifetime utility is the sum or integral of felicity each period. The felicity function depends on consumption, wealth, time, and possibly other state variables such as consumption good prices.

In models with habit formation, past consumption directly affects the felicity derived from future consumption. For example, if consumption falls from a high to a lower level, regret may make the contribution to felicity smaller than it would have been without the previous prosperity. In this case, one state variable of the felicity function might be last period's consumption or an average of consumption over several past periods.

In a deterministic setting, future consumption could also serve as one or more state variables in a completely analogous fashion. For example, the anticipation of future high consumption might enhance (or worsen) the enjoyment of current consumption. However, this is not practical in a stochastic model; the state variables must be known to the agent at the evaluation time. While future consumption cannot be a state variable, expected future consumption, the variance of future consumption, or other measurable aspects of anticipated consumption could be so used. In this paper, current post-consumption wealth[3] will be used as the primary state variable. Since wealth is the present value of future consumption, it clearly is some measure of future anticipated consumption. Furthermore, it has the considerable practical advantage of

[3] Either post- or pre-consumption wealth can be used as a state variable with no loss of generality. Knowing consumption plus either one is sufficient to determine the other. Choosing post-consumption is more in keeping with notion that wealth is an indicator of future consumption and makes some of the following discrete-time modeling easier.

being much more readily observable than many other candidates. This view of felicity is similar to that in the expectations-based reference-dependent utility model of Kőszegi and Rabin (2006) though here the addition to lifetime utility comes from the anticipation rather than the change in anticipation.

Alternatively, it could be argued that being wealthy by itself simply provides enjoyment above and beyond any consumption it allows so wealth contributes directly to felicity. However, we do not assume here that felicity must be increasing in wealth holding consumption fixed. Higher wealth that is not currently being spent on consumption could instead lead to frustration that lowers felicity just as high past consumption leads to regret.

Another interpretation is that fluctuations in wealth provide an indication of changes that affect the economy in the long-term. For example, wealth will drop if a worsening of investment prospects lowers the growth rate of consumption. Obviously this might affect an investor's felicity even if consumption did not change — for example in a pure exchange model. Using wealth in the felicity function enables modeling concerns about long-term risk in a direct fashion.

In addition to capturing the effects discussed above, another reason for using wealth in the felicity function is to proxy for the effects of omitted variables. A change in the investment or consumption opportunity set will typically affect the consumption-savings decision so measuring both wealth and consumption might capture such changes if they are difficult to observe. For example, consumption is typically measured as the total value of the various good consumed. But felicity cannot in general be measured correctly this way. An investor who perceives that the relative prices of consumption goods may change will alter his saving decision. Measuring felicity by the total expenditure each period will misstate his lifetime utility, but adding wealth to the felicity function can help to correct this measurement error.[4]

The basic model employed in this paper structures lifetime utility as the sum $U = \sum u(C_t, W_t, \mathbf{x}_t, t)$ or the corresponding integral. In implementation, it will often be assumed that time is relevant only as a subjective discounting factor and that felicity is otherwise state-independent.

This utility function provides a measure for the agent to compare any and all lifetime packages of wealth and consumption regardless of their feasibility or optimality. This is, of course, true for standard utility of consumption functions as well; it is only pointed out here to pre-empt any confusion that the wealth-dependent felicity function is an indirect utility function. Indirect or derived utility, $J(W, \mathbf{x}, t)$, is determined here in the usual fashion as the present expected value of future felicity; it does not depend on consumption — only on wealth, time, and the other state variables.

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[4] For example, consider an investor with utility for two goods, $u(c_1, c_2) = (c_1^{\gamma} c_2^{1-\gamma})^{\alpha}/\alpha$ Let good one be the numerarire and p be the price of good two. From standard price theory, a total consumption expenditure of C will be allocated as $c_1 = \gamma C$, $c_2 = (1-\gamma)C/p$ giving utility of $AC^{\alpha} p^{(\gamma-1)\alpha}$. For this power felicity function, total consumption expenditure will be proportional to wealth $C = f(p)W$ where f may depend on time and the investment opportunity set as well as the price of good two. After solving for p in terms of C/W, utility can be re-expressed as a function of total consumption expenditure and wealth.

Because, with respect to consumption alone, lifetime utility is time additive, the intertemporal elasticity of substitution of consumption (EISC) will be related to the inverse of the risk aversion over consumption. That is, for $U = \cdots + u^t(C,W) + \cdots \; + u^\tau(c,w) + \cdots,$ the reciprocal of the EISC is[5]

$$\frac{1}{\text{EISC}} \equiv \frac{d[\ell\text{n}(U_C/U_c)]}{d[\ell\text{n}(c/C)]} = \frac{d(U_C/U_c)}{d(c/C)}\frac{c/C}{U_C/U_c} = \frac{Cu_C^t(-cu_{cc}^\tau/u_c^\tau) + cu_c^\tau(-Cu_{CC}^t/u_C^t)}{Cu_C^t + cu_c^\tau}. \tag{1}$$

which is a weighted average of the relative risk aversion of consumption in the two periods. However, investment decisions will also depend on the risk-aversion about wealth and the joint risk aversion about wealth and consumption.

III Individual-Investor Consumption-Wealth Pricing

Suppose an investor has a time-additive utility function where the felicity each period depends on consumption and post-consumption wealth but is otherwise state independent. Lifetime utility is

$$\sum u(C_t, W_t, t). \tag{2}$$

Assume that an optimal lifetime plan exists for this investor. Now consider an alteration to this plan that reduces consumption by a small amount ε at time t which is invested in asset n. Whatever this extra investment grows to by time $t+1$ is then consumed along with the original optimally planned consumption. That is, the lifetime plan is altered only at times t and t+1, and post-consumption wealth is changed only at time t. To first order in the deviation, ε, the change in expected lifetime utility is

$$\Delta\mathbb{E}\sum u(C_t, W_t, t) = \left(-u_C(C_t, W_t, t) + u_W(C_t, W_t, t) + \mathbb{E}[u_C(\tilde{C}_{t+1}, \tilde{W}_{t+1}, t)\tilde{R}_{n,t+1}]\right)\varepsilon + o(\varepsilon) \tag{3}$$

where $\tilde{R}_{n,t+1}$ is one plus the rate of return earned on asset n over the period t to t+1.

The change in expected lifetime utility given in (3) must be zero if the original plan was optimal; therefore

$$1 = \frac{\mathbb{E}[u_C(\tilde{C}_{t+1}, \tilde{W}_{t+1}, t+1)\tilde{R}_{n,t+1}]}{u_C(C_t, W_t, t) - u_W(C_t, W_t, t)}. \tag{4}$$

This relation is almost identical to the standard time-additive result; the difference being the additional term in the denominator. The result holds for all assets including the risk-free asset

[5] This result can be derived with the substitutions

$$d(U_C/U_c) = \frac{\partial(U_C/U_c)}{\partial C}dC + \frac{\partial(U_C/U_c)}{\partial c}dc = \frac{U_{CC}/U_c - U_C U_{Cc}}{U_c^2}dC + \frac{U_{Cc}/U_c - U_C U_{cc}}{U_c^2}dc$$

$$d(c/C) = (Cdc - cdC)/C^2 \qquad\qquad dc = -(U_C/U_c)dC$$

where the last equality is the definition of an isoquant of constant utility.

(with gross return R_f) and the term in the denominator is the same for all assets so for any asset

$$\mathbb{E}[u_C(\tilde{C}_{t+1}, \tilde{W}_{t+1}, t+1)(\tilde{R}_{n,t+1} - R_f)] = 0 . \tag{5}$$

Expressing the expectation of the product using covariance, equation (5) can be rewritten in a more familiar fashion as

$$\mathbb{E}[\tilde{R}_{n,t+1} - R_f] = \frac{\operatorname{cov}[u_C(\tilde{C}_{t+1}, \tilde{W}_{t+1}, t+1), \tilde{R}_{n,t+1}]}{\operatorname{cov}[u_C(\tilde{C}_{t+1}, \tilde{W}_{t+1}, t+1), \tilde{R}_{p,t+1}]} \mathbb{E}[\tilde{R}_{p,t+1} - R_f] \tag{6}$$

where p denotes any convenient portfolio with a non-zero risk premium.[6]

As an illustration, suppose the tradeoff between current consumption and wealth is governed by an ordinal Cobb-Douglas function and risk aversion is determined by a power utility function; i.e., $u(C, W, t) = e^{-\delta t}(C^{1-\theta} W^{\theta})^{\alpha}/\alpha$ with $\alpha < 1$, $\theta \le 1$, $\alpha(1-\theta) < 1$.[7] The relation is then

$$\mathbb{E}[\tilde{R}_{n,t+1} - R_f] = \frac{\operatorname{cov}[\tilde{C}_{t+1}^{\alpha(1-\theta)-1} \tilde{W}_{t+1}^{\alpha\theta}, \tilde{R}_{n,t+1}]}{\operatorname{cov}[\tilde{C}_{t+1}^{\alpha(1-\theta)-1} \tilde{W}_{t+1}^{\alpha\theta}, \tilde{R}_{p,t+1}]} \mathbb{E}[\tilde{R}_{p,t+1} - R_f] . \tag{7}$$

Compare this CAPM-style equation to that derived by Epstein and Zin (1991). They assumed a representative agent with recursive utility of consumption

$$v_t = [C_t^{\rho} + \beta(\mathbb{E}[\tilde{v}_{t+1}^{\alpha} \mid W_t, \mathbf{x}_t])^{\rho/\alpha}]^{1/\rho} \tag{8}$$

with a constant subjective discount factor, β, whose aggregator has a constant intertemporal elasticity of substitution, $(1-\rho)^{-1}$, and whose relative risk aversion is constant, $1-\alpha$. They derived a representative-investor result in their equation (15) that can be expressed as

$$\mathbb{E}[\tilde{R}_{n,t+1} - R_f] = \frac{\operatorname{cov}[\tilde{C}_{t+1}^{\alpha(\rho-1)/\rho} \tilde{M}_{t+1}^{\alpha/\rho-1}, \tilde{R}_{n,t+1}]}{\operatorname{cov}[\tilde{C}_{t+1}^{\alpha(\rho-1)/\rho} \tilde{M}_{t+1}^{\alpha/\rho-1}, \tilde{R}_{p,t+1}]} \mathbb{E}[\tilde{R}_{p,t+1} - R_f] \tag{9}$$

where M is the return on the market portfolio. Assuming a representative investor in both cases,

[6] For standard time-additive utility, this same result holds if the extra is consumed in any later period so equations like (4) through (6) also hold for multi-period returns, and $1 = \mathbb{E}[u_C(\tilde{C}_\tau, \tau)\tilde{R}_{\tau,t}]/u_C(C_t, t)\ \forall \tau > t$. The analogous relation is not true for consumption-wealth felicity since wealth and therefore felicity is also changed in each intervening period. This one-period restriction also applies to recursive utility which is built on a one-period-ahead comparison.

[7] The requirements are: (i) $\theta < 1$ to keep the marginal utility of consumption positive, (ii) $\alpha < 1$ for the investor to be risk averse and have a well-defined optimal portfolio, and (iii) $\alpha(1-\theta) < 1$ so that the investor is risk averse about consumption. This final requirement is more esthetic than strictly necessary. Finite optimal plans are possible in some cases when (iii) is violated; see footnote 10. Typically $\theta > 0$ is required for Cobb-Douglas utility, but it need not be assumed that more wealth is preferred *when consumption is held constant*. If the anticipation of higher future consumption (i.e., high current wealth) reduces the felicity of current consumption, then θ will be negative. Standard consumption-only felicity is $\theta = 0$.

we can see that (7) and (9) are quite similar for $\rho = \alpha/(1+\alpha\theta)$. The difference is that Epstein-Zin preferences use the return on the market portfolio from t to $t+1$ which includes time $t+1$ consumption while Cobb-Douglas consumption-wealth felicity uses post-consumption wealth at time t+1. In the continuous-time limit, this distinction is removed.

Two special cases that tie the models together should be pointed out. First, when $\alpha = \rho$, the Epstein-Zin model reduces to the standard time-additive utility model with constant relative risk aversion of $1-\alpha$. This obviously corresponds to $\theta = 0$ in the consumption-wealth felicity model. Second, with an infinite elasticity of consumption ($\rho = 1$), only the market portfolio remains in the equilibrium relation in (9). This corresponds to $\alpha(1-\theta) = 1$ in the consumption-felicity model in which case felicity is linear in consumption so the investor is neutral towards consumption risk (but not wealth risk).

Other functional forms can be used as well in the general result given in (6). Most of them, of course, lead to expressions more cumbersome than (7). Regardless, the resulting discrete-time relation is typically linearized before being applied to any data. The need for linearization can be avoided entirely by working in continuous time as shown below.

In continuous-time, this result can be expressed with the martingale pricing relation. Assuming felicity is otherwise state-independent and that prices, consumption, and wealth follow diffusion processes, we have for the price P of any portfolio or asset with its dividends reinvested

$$\begin{aligned}
0 = \mathbb{E}[d(u_C(C,W,t)P)] &= P\mathbb{E}[u_C dP/P + u_{Ct}dt + u_{CC}dC + \tfrac{1}{2}u_{CCC}dC^2 + u_{CW}dW \\
&\qquad + \tfrac{1}{2}u_{CWW}dW^2 + u_{CCW}dCdW + u_{CC}dCdP/P + u_{CW}dWdP/P] \\
&= P[u_C\mu_P + u_{Ct} + u_{CC}C(\zeta_C + \sigma_{PC}) + \tfrac{1}{2}u_{CCC}\sigma_C^2 C^2 \\
&\qquad + u_{CW}W(\zeta_W + \sigma_{PW}) + \tfrac{1}{2}u_{CWW}\sigma_W^2 W^2 + u_{CCW}CW\sigma_{CW}]dt
\end{aligned} \tag{10}$$

where ζ_i and σ_{ij} denote the (not necessarily constant) expected rates of growth and covariances between the growth rates of the items. The differences between (10) and the standard model are the final three terms with the cross partial derivatives.

For the risk free asset, $\mu_P = r$ and $\sigma_{PC} = \sigma_{PW} = 0$, so (10) gives an interest rate of

$$r = -\frac{u_{Ct}}{u_C} - \frac{Cu_{CC}}{u_C}\zeta_C - \frac{C^2u_{CCC}}{2u_C}\sigma_C^2 - \frac{Wu_{CW}}{u_C}\zeta_W - \frac{W^2u_{CWW}}{2u_C}\sigma_W^2 - \frac{CWu_{CCW}}{u_C}\sigma_{WC}\,. \tag{11}$$

Again, the final three terms are new for consumption-wealth felicity. For any other asset or portfolio, the risk premium is

$$\mu_P - r = -\frac{Cu_{CC}}{u_C}\sigma_{PC} - \frac{Wu_{CW}}{u_C}\sigma_{WP}\,. \tag{12}$$

For the same relative risk aversion over consumption, the risk premium on a portfolio positively correlated with wealth will be larger (smaller) under consumption-wealth felicity if the cross second derivative, u_{CW}, is negative (positive).[8] The effect of consumption-wealth felicity on the interest rate is harder to assess, but, again, if felicity is additively separable in consumption and wealth, there is no additional effect.[9]

The simple intuition for the risk premium and interest rate can be given using the Cobb-Douglas felicity function from the discrete-time example above because all of the utility ratios in (11) and (12) are then constant. Three parameters determine these utility ratios, δ, α, and θ. The first, δ, represents pure time preference. The second, α, measures the investor's over-all risk aversion. If consumption and wealth are scaled by k at every point in time, then utility rises by a factor of k^{α} just as it does for the power utility function C^{α}/α, so the investor's relative risk aversion for scaling up or down his entire optimal plan is $1-\alpha$. The third parameter, θ, measures the trade-off between wealth and consumption. Somewhat more intuitive measures are $\chi \equiv -Cu_{CC}/u_C = 1-\alpha+\alpha\theta$ and $\varphi \equiv -\alpha\theta$. The first, χ, is the relative risk aversion for consumption alone, or, equivalently since utility is additive, the reciprocal of the EISC. The second, $\varphi = 1-\alpha-\chi$, is the difference between relative risk aversion and the reciprocal of EISC; this is a measure of the deviation from simple time-additive utility of consumption.

Using these utility parameters, the risk premium in (12) is

$$\mu_P - r = \chi\sigma_{PC} + \varphi\sigma_{PW} \;. \tag{13}$$

This equation nicely illustrates the equity premium puzzle. For felicity of consumption alone, relative risk aversion and the reciprocal of the EISC are equal ($\varphi = 0$) so the asset's covariance with the portfolio of wealth does not affect the risk premium; only its covariance with consumption is important. With consumption-wealth felicity, covariance with wealth does contribute to an asset's risk premium. Note that somewhat paradoxically, covariance with wealth contributes negatively to the risk premium if the reciprocal of the EISC exceeds relative risk aversion ($\varphi < 0$). This condition does not preclude an optimal plan, though it might be presumed unusual.[10]

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[8] The utility measure $-u_{CW}/u_C$ can be given the following interpretation. Consider a gamble which gives simultaneous small shocks, ε_C and ε_W, to both consumption and wealth. The shocks are mean zero with $\mathbb{E}[\varepsilon_C^2] = \sigma_C^2$, $\mathbb{E}[\varepsilon_W^2] = \sigma_W^2$, $\mathbb{E}[\varepsilon_C\varepsilon_W] = \sigma_{CW}$. What is the extra risk premium, π, measured in consumption of this joint risk in excess of the risk premiums, π_C and π_W, of the individual small shocks? This extra premium is defined by $u(C-\pi_C-\pi, W-\pi_W) = \mathbb{E}[u(C+\varepsilon_C, W+\varepsilon_W)]$. Using a Taylor expansion

$$\begin{aligned} u(C,W) - (\pi_C+\pi)u_C - \pi_W u_W &\approx u(C,W) + \tfrac{1}{2}\sigma_C^2 u_{CC} + \tfrac{1}{2}\sigma_W^2 u_{WW} + \sigma_{CW}u_{CW} \\ \Rightarrow \qquad \pi \approx \tfrac{1}{2}\sigma_C^2 \tfrac{-u_{CC}}{u_C} - \pi_C + \tfrac{u_W}{u_C}\left[\tfrac{1}{2}\sigma_W^2 \tfrac{-u_{WW}}{u_W} - \pi_W\right] &+ \tfrac{-u_{CW}}{u_C}\sigma_{CW} = \tfrac{-u_{CW}}{u_C}\sigma_{CW} \;. \end{aligned}$$

So $-u_{CW}/u_C$ is the extra consumption compensation required for such risks per unit of covariance between consumption and wealth. Similarly, the risk measure $-u_{CW}/u_W$ can be shown to give the extra amount of wealth that would compensate for this covariance risk.

[9] It bears noting that consumption differs from that in the standard model even when felicity is additively separable in consumption and wealth as shown in section IV.

[10] A similar result is true for Epstein-Zin preferences. The stochastic differential utility expression equivalent to the relation in (9) is $\mu_P - r = [\alpha(1-\rho)/\rho]\sigma_{PC} + [(\rho-\alpha)/\rho]\sigma_{PW}$. Using the stated equivalence between the two models, $\rho = \alpha/(1+\alpha\theta)$, this is identical to (13). Note that somewhat more paradoxically, the covariance with consumption also contributes negatively to the risk premium under Epstein-Zin preferences if α and ρ have opposite signs. This is precluded for consumption-wealth felicity by the esthetic restriction (iii) above. Of course, in either model, if the

Empirically, the variation in consumption is small so a very low EISC (high risk aversion) is required to fit the equity premium when only consumption matters. However, the covariance between equity and wealth would typically be much higher so the consumption-wealth felicity model can help explain the premium.

The interest rate is

$$r = \delta + \chi\zeta_C - \tfrac{1}{2}\chi(1+\chi)\sigma_C^2 + \phi\zeta_W - \tfrac{1}{2}\varphi(1+\varphi)\sigma_W^2 - \chi\varphi\sigma_{WC}\ . \tag{14}$$

The first three terms are standard. The first is pure time preference. The second arises from the desire to smooth consumption intertemporally. The larger the growth rate in consumption and the smaller the EISC, the greater is the desire to borrow against future consumption which leads to a higher interest rate in equilibrium. The third term comes from the demand for precautionary savings with uncertain consumption growth; an investor's precautionary demand pushes the interest rate lower with an effect proportional to consumption's variance. The final three terms are distinctive to this consumption-wealth felicity model. The fourth and fifth terms are similar to the second and third with φ serving the same role for wealth that χ does for consumption (though φ can be positive or negative). If risk aversion exceeds the reciprocal of the EISC ($\varphi > 0$), a higher expected growth of wealth or a lower variance increases the interest rate in equilibrium and conversely if $\varphi < 0$.[11] In addition the consumption-wealth covariance affects the interest rate proportionally to both of these utility ratios.

The two components of the risk premium in time-additive consumption-wealth felicity shown in (13) have the potential to address the equity risk premium puzzle. Even in the simple Cobb-Douglas case, there are two parameters for the calibration whereas Mehra and Prescott (1985) had only one parameter since risk aversion and EISC are linked for time-additive utility. However, fitting the consumption-wealth model is a bit more involved. The subjective discount rate and the correlation between consumption and equity can be ignored when illustrating the puzzle because only a lower bound for the interest rate and an upper bound for the equity premium are required. But consumption and wealth are not highly correlated in the short term so the risk premium should actually be much below this upper bound. Over the Mehra-Prescott period, 1889-1978, the correlation between consumption growth and the return on equity was 22%. Ignoring this correlation in a model based on both consumption and wealth would be unreasonable.

Calibration of the consumption-wealth model also requires an estimate of the growth rate of wealth, ζ_W. This differs from the expected rate of return on the portfolio of wealth by the propensity to consume out of wealth (i.e., $\zeta_W = \mu_W - C/W$). In a pure exchange model of the Lucas type, this difference is just the dividend yield on the wealth portfolio. But in a more general model, it would exclude that portion of the dividends which is reinvested and increases

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investment opportunity set is constant, consumption will be a deterministic fraction of wealth, and any asset's correlations with consumption and wealth will be equal and the risk premiums will reduce to a single factor in both models.

[11]The model does not require that $\alpha\theta < 1$. If $\alpha\theta = -\varphi > 1$, then the investor is risk-seeking over pure wealth gambles (though is still risk averse), and the variance of wealth has the opposite effect on the interest rate.

the growth of wealth. Of course, over the long run, the two growth rates must be equal on average if the economy has a steady state in which the propensity to consume remains away from 0 and ∞.[12]

Fitting equations (13) and (14) to the Mehra-Prescott data (μ_W = 6.98%, σ_W = 16.54%, ζ_C = 1.83%, σ_C = 3.57%, r = 0.80%) and assuming a correlation of 0.22 between wealth and consumption, a wealth growth rate equal to the consumption growth rate of 1.83%, and a subjective discount rate of δ = 1% gives a relative risk aversion of $1-\alpha$ = 5.87 and an EISC of χ^{-1} = 0.264.[13] These numbers are consistent with estimates from various direct studies. This consistency is robust with respect to the assumed correlation with fitted values ranging over $1-\alpha \in (6.02, 5.09)$ and $\chi^{-1} \in (0.266, 0.285)$ for correlations from 0 to 0.9. Similarly if the growth rate of wealth is decreased or increased by up to 50%, $1-\alpha \in (7.33, 4.50)$ and $\chi^{-1} \in (0.188, 0.425)$. Figures 1 and 2 shows this in more detail.

Admittedly, one problem with this calibration is a Roll-type criticism that the portfolio of equities is not the same as the portfolio of wealth. However, this criticism is somewhat muted here for two reasons. First as already noted, the fitted utility parameters are insensitive to the correlation between consumption and wealth. Second, there is no necessity that W represents total wealth in the model, unlike in the CAPM. Rather, W measures whatever affects a consumer's felicity, and this well might be liquid wealth rather than total wealth. All that is required for the derivation above is that whatever W represents can both be used to finance current consumption and be saved for the future by investing in various assets.

Similar results are true for any homogeneous consumption-wealth felicity function, $u(C, W) = W^{\alpha}v(z)$ with $z \equiv C/W$. The risk premium as given in (12) is

$$\mu_P - r = \frac{-zv''}{v'}\sigma_{PC} + \left(1-\alpha+\frac{zv''}{v'}\right)\sigma_{WP} = \chi(z)\sigma_{PC} + [1-\alpha-\chi(z)]\sigma_{WP} \qquad (15)$$

where χ is the reciprocal of the EISC as before. The only difference here is that χ is no longer constant but varies with the ratio of consumption to wealth.

The results presented here hold at a single instant generically — the only requirement is that *some* investor with time-additive consumption-wealth felicity is following an optimal plan. Of course, these relations are useful only if *that* investor's optimal portfolio and consumption

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[12] In fact a reasonable average propensity to consume or its equivalent should be another test of any calibrated model which purports to represent the economy. The equity portfolio used by Mehra and Prescott had an average dividend yield of 4.8%. If we assume this is the propensity to consume a slightly higher wealth growth rate of 2.18% is implied. This estimate leads to calibrated values of $1-\alpha$ = 5.33 and χ^{-1} = 0.310.

[13] There are two calibrated solutions since (14) is quadratic. The separate solutions are

$$\chi = \frac{b \pm \{b^2 + 2(\sigma_C^2\sigma_W^2 - \sigma_{CW}^2)[(r-\delta)\sigma_W^2 - (\mu_W - r)\zeta_W + \tfrac{1}{2}(\mu_W - r)^2]\}^{1/2}}{\sigma_C^2\sigma_W^2 - \sigma_{CW}^2} \qquad \phi = \frac{\mu_W - r - \chi\sigma_{CW}}{\sigma_W^2}$$

$$\text{where} \quad b \equiv \sigma_W^2(\zeta_C - \tfrac{1}{2}\sigma_C^2 + \tfrac{1}{2}\sigma_{CW}) - \sigma_{CW}\zeta_W .$$

The solution with the positive sign is reported here. The other solution, χ^{-1} = 0.040, $1-\alpha$ = 26.05, has a risk aversion much larger than typically assumed.

can be identified. This is much like single-period mean-variance analysis — betas with tangency portfolio always identify risk premiums; the trick is in recognizing the market as the tangency portfolio. In representative-investor models this identification problem is finessed by assuming there is an investor who holds the market portfolio. It is then optimal for someone by assertion, and we need only determine the utility function to use.

In intertemporal models, there is an added complexity. If we wish to use both the market portfolio and per capita consumption in describing returns, the representative investor must be average in both his portfolio and consumption. This requirement is rarely made explicit in representative-investor models. In single-period models, it is innocuous since end-of-period wealth and consumption are equal so an investor holding the market must have per capita consumption as well. Therefore, to show that a representative investor exists we need only examine the aggregated individual portfolio holdings. If the set of efficient (optimal) portfolios is convex under homogeneous beliefs, this aggregate demand will also be an efficient portfolio.[14] The power of Breeden's (1979) intertemporal consumption model is that for time-additive felicity of consumption, equation (12) can be aggregated to price assets using just consumption and ignoring wealth. Unfortunately, this aggregation is generally not possible for either recursive utility or time-additive consumption-wealth utility.

Even if a representative investor exists, we still need to determine his propensity to consume and the volatility of his wealth relative to his consumption to fully compare the model to what is observed. The next sections address these issues by solving the intertemporal consumption-portfolio problem.

IV The Consumption-Portfolio Problem with Consumption-Wealth Felicity

This section solves the consumption-investment problem for individual investors and addresses the problem of aggregation. As in the standard consumption-portfolio problem, the corresponding continuous-time diffusion model has simple, mean-variance results. Unfortunately, aggregation is not typically possible.

Assume that felicity depends on consumption, wealth, and time so lifetime utility is the integral of $u(C_t, W_t, t)$. There are K state variables in a vector, $\mathbf{x}_t$, of information. The N risky assets' prices and the K state variables are jointly Markov and evolve according to diffusions

$$dx_k = \zeta_k(\mathbf{x}_t, t)dt + \xi_k(\mathbf{x}_t, t)d\omega_k \qquad dP_n/P_n = \mu_n(\mathbf{x}_t, t)dt + \sigma_n(\mathbf{x}_t, t)d\omega_{K+n}\,. \tag{16}$$

To simplify the notation and exposition, assume that the state variables are mutually uncorrelated. Since the state variables are unspecified, there is no loss of generality in this assumption as some rotation of them will have this property. The covariance matrices of asset returns is denoted by $\mathbf{\Omega}(\mathbf{x}, t)$ with representative element $\Omega_{nm} = \sigma_n\sigma_m\text{Cov}[\omega_n, \omega_m]$. The covariance matrix of returns on the assets with changes in state variable is $\mathbf{\Lambda}(\mathbf{x}, t)$ with representative element $\Lambda_{nk} \equiv \sigma_n\xi_k\text{cov}[\omega_{K+n}, \omega_k]$. For convenience we assume there is a risk-free asset with an interest rate $r(\mathbf{x}, t)$. In the absence of a risk-free asset, the analysis below remains valid with a "zero-beta"

[14] See Dybvig and Ross (1982) or Ingersoll (1987) for further discussion.

portfolio in its place.

The formulation leading to the Bellman equation is standard and now well-known. At each moment, the investor chooses a consumption flow rate, C, and portfolio, $\mathbf{w}$, to maximize utility. The indirect or derived utility of wealth function, $J(W, \mathbf{x}, t)$, satisfies the standard Bellman equation[15]

$$\begin{aligned} 0 &= \tfrac{1}{2}\mathbf{w}'\boldsymbol{\Omega}\mathbf{w}W^2 J_{WW} + \big([r + \mathbf{w}'(\boldsymbol{\mu} - r\mathbf{1})]W - C\big)J_W + J_t + u(C,W,t) \\ &\quad + \tfrac{1}{2}\sum_k J_{x_k x_k}\xi_k^2 + \sum_k J_{x_k}\zeta_k + \mathbf{w}'\boldsymbol{\Lambda}J_{W\mathbf{x}}W \,. \end{aligned} \tag{17}$$

where $J_{W\mathbf{x}}$ is the vector of partial derivatives $(\partial^2 J/\partial W\partial x_k)$. The first order conditions are

$$\begin{aligned} 0 &= u_C - J_W \\ \mathbf{0} &= W^2 J_{WW}\boldsymbol{\Omega}\mathbf{w} + WJ_W(\boldsymbol{\mu} - r\mathbf{1}) + W\boldsymbol{\Lambda}J_{W\mathbf{x}} \,. \end{aligned} \tag{18}$$

The first of these relations is the standard envelope condition of the optimal lifetime plan; wealth is allocated so that at the margin the last dollar consumed contributes the same marginal utility as the last dollar saved. This equation determines the optimal consumption.

The first and second conditions together show that at the margin each asset must make the same contribution to next-period's expected marginal utility, J_W, from which we get the martingale pricing relation $\mathbb{E}[d(PJ_W)] = 0$.[16] The envelope condition justifies using the marginal utility of consumption in place of the marginal utility of wealth as the stochastic discount factor and circumvents the necessity of solving for the latter.

Solving the second relation gives the individual asset demands as

$$\mathbf{w}^* = \frac{-J_W}{WJ_{WW}}\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - r\mathbf{1}) - \boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\frac{J_{W\mathbf{x}}}{WJ_{WW}} \,. \tag{19}$$

---

[15] See, for example, Ingersoll (1987) or Merton (1990) for a discussion of the standard multi-period portfolio problem. There might be some question why felicity appears as $u(C, W, t)$ rather than $u(C, W - C, t)$. The answer is basically the same as why it is W and not $W - C$ that appears multiplying the mean and variance of the portfolio returns in the J_W and J_{WW} terms. Consumption is a flow rather than a stock like wealth. Financing consumption at the rate C over the interval from t to $t+\Delta t$, requires $C\Delta t$ in wealth leaving $W - C\Delta t$ to invest, but the portfolio's mean and variance are also order Δt so $C\Delta t$ can be ignored. Similarly, once $C\Delta t$ is extracted from wealth to finance consumption over the interval t to $t + \Delta t$, the felicity increment to utility is

$$\int_t^{t+\Delta t} u(C, W - C\Delta t, s)\,ds = u(C, W - C\Delta t, t')\Delta t = u(C, W, t')\Delta t - u_W(C, W, t')C(\Delta t)^2 + o((\Delta t)^2)$$

where $t' \in (t, t+\Delta t)$ by the mean value theorem and the second equality follows from a Taylor expansion. In the limit as $\Delta t \to 0$, the second and higher order terms can be ignored, and $t' \to t$ leaving $u(C, W, t)$.

[16] By Ito's lemma $\mathbb{E}[d(PJ_W)] = \mathbb{E}[J_W dP + PdJ_W + dP \cdot dJ_W] = J_W\mathbb{E}[dP] + P\mathbb{E}[dJ_W] + \mathrm{Cov}[dP, dJ_W]$. Indirect marginal utility grows at the rate $-r$, so $\mathbb{E}[d(PJ_W)] = PJ_W(\mu_P - r)dt + \mathrm{Cov}[dP, dJ_W]$. The right-hand side is zero from the first order condition (18) so PJ_W is a martingale for all asset prices. As the marginal utilities of wealth and consumption are equal, Pu_C is also a martingale as in the standard time-additive utility model. In general the martingale using the direct felicity function is easier to use as the dependence of J on the state variables need not be determined.

The investor's optimal risky-asset portfolio is a combination of the tangency portfolio with weights proportional to $\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu}-r\mathbf{1})$ and K other portfolios with weights proportional to $\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}_k$ where $\boldsymbol{\lambda}_k$ is the k^{th} column of $\boldsymbol{\Lambda}$.[17] The tangency portfolio has the maximum Sharpe ratio giving the best overall trade off between risk and return. Each of the other K portfolios has the maximum possible correlation with one of the state variables[18] and therefore provides the best possible hedge against changes in that state variable. If investors have homogeneous beliefs about the variances and covariances, then these K hedge portfolios are identical for each of them. If they also have homogeneous beliefs about expected returns, then their tangency portfolios are the same as well resulting in a K+1 mutual fund separation result.

In the standard model with time-additive utility of consumption, two equilibrium relations can be derived under homogeneous beliefs: the $K + 1$ factor CAPM and the consumption CAPM. The first is also valid for this model. From (19) each investor, i, forms a portfolio so that

$$a_i(\boldsymbol{\mu}-r\mathbf{1}) = W_i\boldsymbol{\Omega}\mathbf{w}_i^* + b_i\boldsymbol{\Lambda}. \tag{20}$$

When this equation is aggregated across all investors, the first term on the right-hand side is a vector of the covariance of each asset with the market portfolio, and the second term is a matrix of the covariance of each asset with each of the state variables. This leads to the multi-factor CAPM

$$\mu_n - r = \beta_n^M(\mu_M - r) + \sum_k \beta_n^k(\mu_k - r) \tag{21}$$

where the beta coefficients come from the multiple regression of the individual assets on the market and the other K hedge portfolios with holdings in proportion to $\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}$.

The consumption CAPM, however, is not generally valid when felicity depends on wealth. Differentiate the envelope condition $u_C = J_W$ with respect to W and $\mathbf{x}$ to get

$$\begin{aligned} J_{WW} &= \frac{\partial u_C}{\partial W} = \left.\frac{\partial u_C}{\partial C}\right|_{C=C^*}\frac{\partial C^*}{\partial W} + \frac{\partial u_C}{\partial W} \equiv u_{CC}C_W^* + u_{CW} \\ J_{W\mathbf{x}} &= \frac{\partial u_C}{\partial \mathbf{x}} = \left.\frac{\partial u_C}{\partial C}\right|_{C=C^*}\frac{\partial C^*}{\partial \mathbf{x}} \equiv u_{CC}C_{\mathbf{x}}^*. \end{aligned} \tag{22}$$

Substituting into (18) and solving for the excess expected rates of return gives

[17] If there is no risk-free asset, then the optimal portfolio is $\mathbf{w}^* = -(J_W/WJ_{WW})\boldsymbol{\Omega}^{-1}\boldsymbol{\mu} - \boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}(J_{W\mathbf{x}}/WJ_{WW}) + \gamma\boldsymbol{\Omega}^{-1}\mathbf{1}$, where γ is the Lagrange multiplier of the budget constraint. Linear combinations of the portfolios $\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}$ and $\boldsymbol{\Omega}^{-1}\mathbf{1}$ trace out the mean-variance efficient frontier hyperbola. The individual investor results below are the same with the interest rate replaced by the expected rate of return on a portfolio uncorrelated with both the investor's wealth and consumption.

[18] The square of the correlation between any portfolio and state variable x_k is $(\mathbf{w}'\boldsymbol{\lambda}_k)^2(\mathbf{w}'\boldsymbol{\Omega}\mathbf{w})^{-1}\xi_k^{-2}$. Differentiating with respect to $\mathbf{w}$, we have $\mathbf{0} = 2(\mathbf{w}'\boldsymbol{\Omega}\mathbf{w})^{-2}\xi_k^{-2}[(\mathbf{w}'\boldsymbol{\Omega}\mathbf{w})(\mathbf{w}'\boldsymbol{\lambda}_k)\boldsymbol{\lambda}_k - (\mathbf{w}'\boldsymbol{\lambda}_k)^2\boldsymbol{\Omega}\mathbf{w}]$. It is easily verified that any portfolio holding the risky assets in proportion $\boldsymbol{\Omega}^{-1}\boldsymbol{\lambda}_k$ is a solution to this first order condition and provides the maximal possible correlation.

$$\boldsymbol{\mu} - r\mathbf{1} = \frac{-u_{CC}}{u_C}(\boldsymbol{\Omega}\mathbf{w}^* \cdot WC_W^* + \boldsymbol{\Lambda}C_{\mathbf{x}}^*) - \frac{u_{CW}}{u_C} W\boldsymbol{\Omega}\mathbf{w}^* = A\boldsymbol{\psi} + \Gamma W\boldsymbol{\Omega}\mathbf{w}^* \tag{23}$$

$$\text{where} \quad \boldsymbol{\psi} \equiv \boldsymbol{\Omega}\mathbf{w}^* \cdot WC_W^* + \boldsymbol{\Lambda}C_{\mathbf{x}}^* \quad A \equiv -u_{CC}/u_C \quad \Gamma \equiv -u_{CW}/u_C\,.$$

A is the investor's absolute risk aversion over consumption, and Γ is the multivariate wealth-consumption risk aversion. The first term in $\boldsymbol{\psi}$ is the covariance of each asset's return with changes in wealth multiplied by the marginal propensity to consume. The other K terms are the covariance of each asset's return with the change in one of the state variables multiplied by the marginal effect of a change in that state variable on consumption. Therefore, $\boldsymbol{\psi}$ is a vector of the covariance of each asset's return with the change in the investor's optimal consumption. $W\boldsymbol{\Omega}\mathbf{w}^*$ is a vector of the covariance of each asset's return with the investor's optimally invested wealth. Therefore, the risk premium on any asset is equal to the investor's consumption risk aversion multiplied by the covariance of the asset's return with consumption plus wealth-consumption risk aversion multiplied by the covariance of the asset's return with the investor's optimally invested wealth.

In the standard time-additive model, $u_{CW} = 0$ so only the covariance with consumption is relevant. Assuming homogeneous beliefs, allows (23) to be aggregated across investors to give $(\boldsymbol{\mu} - r\mathbf{1})\sum_i A_i^{-1} = \sum_i \boldsymbol{\psi}^i$. The last sum is the vector of covariances of asset returns with aggregate consumption so each asset's risk premium is proportional to its covariance with aggregate or per capita consumption. As this relation holds for all assets and portfolios, the well-known consumption-CAPM relation is obtained.

$$\mu_n - r = \frac{\sigma_{nC}}{\sigma_{PC}}(\mu_P - r) \tag{24}$$

where P denotes any portfolio with a non-zero correlation with consumption, and σ_{nC} and σ_{PC} are covariances of the n^{th} asset and portfolio P with per capita consumption.

If felicity depends on wealth as well as consumption, then the result in (24) can be derived from (23) in general only if felicity is a linearly separable function of consumption and wealth, i.e., $\Gamma = 0$. In general, when $\Gamma \neq 0$, aggregation of equation (23) does not lead to such simple results. If we divide by A_i before aggregating and solve for risk premiums, we obtain

$$\boldsymbol{\mu} - r\mathbf{1} = \left(\sum_i A_i^{-1}\right)^{-1}\left[\sum_i \boldsymbol{\psi}_i - \boldsymbol{\Omega}\sum_i \mathbf{w}_i W_i \Gamma_i A_i^{-1}\right]. \tag{25}$$

The first term is covariance with aggregate consumption as before; unfortunately, the market portfolio is $\sum_i W_i \mathbf{w}_i$ so unless the ratio Γ_i / A_i is the same for all investors, the aggregated portfolio in the second term will only coincidentally be the market portfolio.[19] Alternatively, if we divide by Γ_i before aggregating we get

[19] One simple case in which this aggregation is possible for heterogeneous investors is for the felicity function $u_i(C, W) = -\exp[-a_i(t)(C + bW)]$. In this case, $A_i = a_i$ and $\Gamma_i = ba_i$ so the ratio $\Gamma_i/A_i = b$ for all investors, and risk premia can be expressed as the sum of the covariances with the market and per capita consumption. This model is examined further in the next section.

$$\boldsymbol{\mu} - r\mathbf{1} = \left(\sum_i \Gamma_i^{-1}\right)^{-1}\left[\sum_i \boldsymbol{\psi}_i A_i \Gamma_i^{-1} - \boldsymbol{\Omega}\sum_i \mathbf{w}_i W_i\right]. \tag{26}$$

The second term is now the vector of covariances with the market portfolio, but the first term is a vector of covariances with some weighted sum of individual consumptions that will typically differ from aggregate consumption.

Stated more simply, when investors have consumption-wealth felicity, there is no guarantee that the average consumer is also the average investor so that equation (23) cannot necessarily be aggregated to give both per capita consumption and the market portfolio as the relevant combinations. Of course if we assume a representative investor who is average in both consumption and portfolio holdings, then the portfolio **w*** is the market and $\boldsymbol{\psi}$ is the vector of covariances of each asset with per capita consumption with no aggregation required.

With a representative investor, the utility terms in (23) can be eliminated by using the risk premiums on the market and any other portfolio (with a non-zero risk premium). Premultiply (23) by the weights of the market and some other portfolio, *P*, and solve for the utility terms. Then the equilibrium can be expressed as

$$\begin{aligned}\mu_n - r &= \frac{\sigma_{nM}\sigma_{PC} - \sigma_{nC}\sigma_{MP}}{\sigma_M^2\sigma_{PC} - \sigma_{MC}\sigma_{MP}}(\mu_M - r) + \frac{\sigma_{nC}\sigma_M^2 - \sigma_{nM}\sigma_{MC}}{\sigma_M^2\sigma_{PC} - \sigma_{MC}\sigma_{MP}}(\mu_P - r) \\ &= \beta_n^{M,C}(\mu_M - r) + \beta_n^{P,C}(\mu_P - r).\end{aligned} \tag{27}$$

The two betas will be recognized as the multiple regression coefficients of asset *n*'s return on the market and portfolio *P* using per capita consumption as an instrumental variable for the latter. If we choose that portfolio, *P**, whose return is most highly correlated with changes in per capita consumption, we do not need the instrumental variable. The maximally-correlated portfolio is $\mathbf{w}_{P*} = k\boldsymbol{\Omega}^{-1}\boldsymbol{\psi}$ where *k* is the normalizing constant.[20] The vector of covariances between this portfolio and each asset is directly proportional to the vector of covariances between per capita consumption and each asset, $(\sigma_{nP*}) = \boldsymbol{\Omega}\mathbf{w}_{P*} = k\boldsymbol{\psi} = k(\sigma_{nC})$. Eliminating the factor of $1/k$ in each term of the numerator and denominator of both betas gives

$$\beta_n^{M,C} = \frac{\sigma_{nM}\sigma_{P*}^2 - \sigma_{nP*}\sigma_{MP*}}{\sigma_M^2\sigma_{P*}^2 - \sigma_{MP*}^2} = \beta_n^M \qquad \beta_n^{P*,C} = \frac{\sigma_{nP*}\sigma_M^2 - \sigma_{nM}\sigma_{MP*}}{\sigma_M^2\sigma_{P*}^2 - \sigma_{MP*}^2} = \beta_n^{P*} \tag{28}$$

which are the usual OLS betas without an instrument. This formulation obviates the need to determine the covariances between assets and consumption; it is, however, subject to a Roll-type criticism of requiring the identification of a very specific portfolio.

The result in (23) and the following analysis is identical to that derived by Duffie and Epstein (1992, eqn. 21) using stochastic differential utility. This correspondence mimics that between the discrete-time equilibrium given in (7) and the Epstein and Zin (1993) recursive-utility model. In each case the diffusion processes have simplified the expression to a mean-

[20] The verification that $\boldsymbol{\Omega}^{-1}\boldsymbol{\psi}$ is the portfolio of risky assets with maximum correlation with per capita consumption is identical to that shown in footnote 18 for the hedge portfolios.

variance one. Note, however, that the derivation using stochastic differential utility requires that the representative investor's value function and optimal consumption function be collinear in the state variables, **x**. This is true under homothetic preferences like those of Epstein-Zin or trivially when the investment opportunity set is constant. But both of these functions are endogenous so it is difficult to state general exogenous conditions for which they hold.

The close connection between stochastic differential utility and consumption-wealth felicity can be seen in general by comparing the Bellman equations for the two problems. Let $f(C,\mathcal{J},t)$ be the normalized aggregator[21] for the value function, $\mathcal{J}(W,\mathbf{x},t)$ in the stochastic differential utility problem. Under the assumptions given above, the Bellman equation for $\mathcal{J}$ is

$$
\begin{aligned}
0 = \tfrac{1}{2}\mathbf{w}'\mathbf{\Omega}\mathbf{w}W^2\mathcal{J}_{WW} &+ \big([r+\mathbf{w}'(\boldsymbol{\mu}-r\mathbf{1})]W - C\big)\mathcal{J}_W + \mathcal{J}_t + f(C,\mathcal{J},t) \\
&+ \tfrac{1}{2}\sum_k \mathcal{J}_{x_k x_k}\xi_k^2 + \sum_k \mathcal{J}_{x_k}\zeta_k + \mathbf{w}'\mathbf{\Lambda}\,\mathcal{J}_{W\mathbf{x}}W\,.
\end{aligned}
\tag{29}
$$

as shown by Duffie and Epstein (1992a). This equation is identical in form to (17) if we assume a state-dependent felicity of consumption-wealth function $u(C,W,\mathbf{x},t)$ = $f(C,\mathcal{J}(W,\mathbf{x},t),t)$. However, exactly which aggregator corresponds to which consumption-wealth felicity function remains to be determined.

V Consumption-Wealth Felicity with Constant Relative Risk Aversion

Models of time-additive utility with constant relative risk aversion are used widely in finance because they are among the few functions that allow closed-form solutions to the dynamic programming problem. For time-additive power utility, consumption is proportional to wealth and holdings of all assets scale proportionally to wealth. These two properties also arise in other utility structures. For example, recursive utility with Epstein-Zin preferences shares the properties of power utility. Consumption-wealth felicity can also have constant relative risk aversion as we examine in this section.

Constant relative risk aversion is particularly useful in representative-agent models since it leads to stationary results that can be easily calibrated. Consumption-wealth felicity displays a type of constant relative risk aversion of $1-\alpha$ if the felicity function is homogeneous of degree α in wealth and consumption. Such a felicity can be expressed as $u(C, W, t) = W^{\alpha}V(z,t)$ where $z \equiv C/W$. We assume that $\alpha < 1$ and that V is twice differentiable in z with $\partial^2 V/\partial z^2 < 0$ and once differentiable in t. These assumptions assure the investor is risk averse in his portfolio choice.[22]

It is not required that felicity be monotone increasing in wealth or consumption though

[21] Stochastic differential utility uses an aggregator and a variance multiplier; these correspond approximately to the aggregator and certainty equivalent function of Kreps-Porteus (1978) discrete-time utility. As shown by Duffie and Epstein (1992a), there exists for each aggregator-multiplier pair an ordinally equivalent normalized pair for which the variance multiplier is zero. This equivalent pair is the normalized aggregator.

[22] Two generalizations can easily be accommodated. Felicity can be state-dependent, $u(C, W, \mathbf{x}, t) = W^{\alpha}V(z, \mathbf{x}, t)$, with results similar to those presented here. Also a utility of bequest in the form $B(\mathbf{x},T)W_T^{\alpha}$ can be added as in the standard model.

two partial restrictions are sensible. Felicity should be increasing in wealth holding z constant; that is, a proportional increase in both wealth and consumption should lead to greater felicity; therefore, $\alpha V > 0$. Felicity should also be increasing in consumption when z is small otherwise the investor will save his entire wealth.[23] Some additional restrictions on V are necessary to avoid a transversality violation in the infinite horizon problem.

For homogeneous felicity, indirect utility of wealth has the form $J(W, \mathbf{x}, t) = W^\alpha G(\mathbf{x},t)/\alpha$ which has a constant relative risk aversion of $1-\alpha$. This claim can be established by substituting the assumed functional form along with the optimal policies from (18) into the Bellman equation giving

$$
\begin{aligned}
0 = W^\alpha G(\mathbf{x},t)\Bigg[& \frac{1}{2(1-\alpha)}\left(S^2 + \frac{2\mathbf{s}' G_\mathbf{x}}{G} + \frac{G_{\mathbf{x}'}\mathbf{S}G_\mathbf{x}}{G^2}\right) + r - V_z^{-1}(G,t) \\
& + \frac{G_t}{\alpha G} + \frac{V\left(V_z^{-1}(G,t),t\right)}{G} + \frac{1}{2\alpha}\sum_k \frac{G_{x_k x_k}}{G}\xi_k^2 + \frac{1}{\alpha}\sum_k \frac{G_{x_k}}{G}\zeta_k \Bigg]
\end{aligned}
\tag{30}
$$

where $S^2 \equiv (\boldsymbol{\mu} - r\mathbf{1})'\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - r\mathbf{1})$, $\mathbf{s} \equiv \boldsymbol{\Lambda}'\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu} - r\mathbf{1})$, and $\mathbf{S} \equiv \boldsymbol{\Lambda}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}$.

Since the terms in brackets do not depend on consumption or wealth, but only on the state variables and time, they can be solved to determine $G(\mathbf{x}, t)$. This verifies the assumed form of J.

As given by the first-order conditions in (18), optimal consumption is proportional to wealth and the optimal portfolio is independent of wealth[24]

$$
C^* = V_z^{-1}\left(G(\mathbf{x},t)\right)W \qquad \mathbf{w}^* = \frac{\boldsymbol{\Omega}^{-1}(\mathbf{x})\left[(\boldsymbol{\mu}(\mathbf{x}) - r(\mathbf{x})\mathbf{1}) + \boldsymbol{\Lambda}(\mathbf{x})G_\mathbf{x}/G\right]}{1-\alpha}. \tag{31}
$$

For this felicity, relative risk aversion of consumption is $-zV''/V'$ which need not be constant; however, portfolio choice is based on the indirect utility function which does display constant relative risk aversion of $1-\alpha$ so the optimal risky-asset portfolio is independent of wealth. In addition consumption is proportional to wealth just as with time-additive power utility.

---

[23] Since indirect utility is increasing in wealth and the envelope condition, $u_C = J_W$, holds, the marginal utility of consumption must be positive at the optimum. Furthermore, since V is concave in z, the marginal utility of consumption must be positive at any lower consumption as well.

[24] V should be increasing in z for small values but need not be increasing for large z. That is, the investor may have a wealth-dependent satiation point, $\bar{C} = \beta(t)W$, above which felicity decreases due to anticipated lower consumption in the future. In this case $V_z^{-1}(G,t)$ is the inverse function of $\partial V/\partial z$ determining z for $z < \beta(t)$. Since V is strictly concave, it is decreasing for $z > \beta(t)$, and optimal consumption will never exceed $\beta(t)W$ as this would decrease both felicity and indirect utility. One example is the felicity function $e^{-\delta t}W^\alpha(z - \beta(t))^2/(2\alpha)$ with $\alpha < 0$. Optimal consumption is $[\beta(t) + \alpha G(\mathbf{x},t)]W < \beta(t)W$. Indirect utility, $J(W,\mathbf{x},t) = e^{-\delta t}G(\mathbf{x},t)W^\alpha/\alpha$, is strictly increasing in W so the investor is never wealth-satiated.

V.1 Constant Relative Risk Aversion with a Static Opportunity Set

As in the standard Merton (1969, 1971) framework, solving the portfolio problem typically requires numerical methods unless the investment opportunity set is constant. With a constant opportunity set, consumption is a deterministic though possibly time-dependent fraction of wealth and the optimal portfolio is the mean-variance efficient tangency portfolio with some leverage.

For example, consider the Cobb-Douglas felicity function, $u = e^{-\delta t} C^{\alpha(1-\theta)} W^{\alpha\theta}/\alpha$. It can be readily verified that derived utility of wealth and optimal consumption are[25]

$$J(W,t) = e^{-\delta t}(1-\theta) H^{1-\alpha(1-\theta)}(t) W^{\alpha}/\alpha \qquad C^* = \frac{W}{H(t)}$$
$$\text{where} \quad H(t) \equiv \frac{1}{z_{\text{CD}}}[1 - e^{-z_{CD}(T-t)/(1-\theta)}] \qquad z_{\text{CD}} \equiv \frac{1-\theta}{1-\alpha(1-\theta)}\left[\delta - \alpha r - \frac{\alpha S^2}{2(1-\alpha)}\right]. \tag{32}$$

This is very similar to the optimal consumption under standard time-additive power utility as in Merton or Epstein-Zin preferences where consumption is

$$C^* = W z_i [1 - e^{-z_i (T-t)}]^{-1}$$
$$\text{with} \quad z_{\text{Merton}} \equiv \frac{1}{1-\alpha}\left[\delta - \alpha r - \frac{\alpha S^2}{2(1-\alpha)}\right] \qquad z_{\text{EZ}} \equiv \frac{1}{1-\rho}\left[\delta_{\text{EZ}} - \rho r - \frac{\rho S^2}{2(1-\alpha)}\right]. \tag{33}$$

In each case z_i is the asymptotic long-term propensity to consume; i.e., the constant propensity of an infinitely-lived investor.[26]

Comparing Cobb-Douglas consumption to that in the standard time-additive utility model

$$z_{\text{CD}} = \frac{(1-\theta)(1-\alpha)}{1-\alpha(1-\theta)} z_{\text{Merton}} . \tag{34}$$

---

[25] This problem can be solved by substituting the trial solution into the Bellman equation. The function H is then the solution to the ordinary first-order differential equation with constant coefficients

$$0 = \left[\tfrac{1}{2}(1-\alpha)^{-1}\alpha S^2 + \alpha r - \delta\right] H(t) + [1-\alpha(1-\theta)]H'(t) + (1-\theta)^{-1}[1-\alpha(1-\theta)] .$$

[26] As can be inferred from (34) below, the transversality condition of the Cobb-Douglas consumption-wealth felicity problem is the same as that for the standard Merton problem. The optimal plan, if one exists, has a constant z and holds a portfolio levered to a risk premium of $S^2/(1-\alpha)$ and standard deviation of $S/(1-\alpha)$. Expected lifetime utility is

$$\alpha^{-1}\int_0^\infty e^{-\delta t}\mathbb{E}[z^{\alpha(1-\theta)} W_t^{\alpha}]dt = \alpha^{-1} z^{\alpha(1-\theta)} W_0^{\alpha} \int_0^\infty \exp\left([-\delta + \alpha(\zeta_W - \tfrac{1}{2}\sigma_W^2) + \tfrac{1}{2}\alpha^2\sigma_W^2]t\right) dt$$
$$= \alpha^{-1} z^{\alpha(1-\theta)} W_0^{\alpha} \int_0^\infty \exp\left([-\delta + \alpha r + \tfrac{1}{2}\alpha(1-\alpha)^{-1}S^2 - \alpha z]t\right) dt .$$

Unless $\delta > \alpha r + \tfrac{1}{2}\alpha S^2/(1-\alpha)$, the integral is divergent for $z = 0^+$.

The ratio $z_{\mathrm{CD}}/z_{\mathrm{Merton}}$ is decreasing in θ and equal to one when θ = 0 so the investor with Cobb-Douglas felicity consumes at a slower (faster) rate than a standard investor with the same risk aversion when θ is positive (negative). The intuition is clear. In the standard model wealth does not contribute directly to felicity, but here the marginal felicity of wealth[27] is positive (when $\theta > 0$) so less is consumed to keep wealth at a higher level than that desired by an ordinary investor. For $\theta < 0$, the opposite is true.

In the standard CRRA consumption-felicity model, the single risk aversion parameter determines not only the interest rate and the risk premium, as has been made clear in the equity premium puzzle literature, but also the average propensity to consume. This number has not received the same attention as the other two, but it is another refutable prediction of these models. In the consumption-felicity model, the optimal consumption rate is not pegged by risk aversion, but can take on a wide range of values. In the Cobb-Douglas case, the parameter θ determines how consumption differs from the time-additive case. The parameter θ must be less than $1 - 1/\alpha$. If the investor has a relative risk aversion greater than or equal to one ($\alpha \le 0$), then θ can take on any value less than one, and optimal consumption falls from $(\alpha-1)z_{\mathrm{Merton}}/\alpha$ to 0 as θ increased from $-\infty$ to 1. If the investor has a relative risk aversion less than one ($\alpha > 0$), then the constraint is binding, and optimal consumption falls from ∞ to 0 as θ increases from its lower limit to 1. This is shown in more detail in Figure 3.

Comparing the consumption rate to that of an Epstein-Zin investor, we see that the $z_{\mathrm{CD}} = z_{\mathrm{EZ}}$ for $\rho = (1-\theta)\alpha$ and $\delta_{\mathrm{EZ}} = (1-\theta)\delta$. Note that the first relation equates the EISC in both models. Of course this matching is only valid for infinitely-lived investors. The extra factor of $1-\theta$ in the exponent in (32) means that originally equal consumption rates with change differently as time passes.

Many additional models can be solved for infinitely-lived CRRA investors with constant rates of time preference, i.e., $V(z, t) = e^{-\delta t} v(z)$. Such investors have indirect utility functions $J(W, t) = e^{-\delta t} v'(z^*) W^{\alpha}/\alpha$ and optimal consumption as a constant fraction of wealth $z^* = C^*/W$ satisfying[28]

$$\frac{v(z^*)}{v'(z^*)} - z^* = \frac{\delta}{\alpha} - r - \frac{S^2}{2(1-\alpha)} = \frac{1-\alpha}{\alpha} z_{\mathrm{Merton}}\ . \tag{35}$$

V.2 Constant Relative Risk Aversion with a Stochastic Opportunity Set

When the investment opportunity set is non-stochastic, consumption is a deterministic fraction of wealth; so the two are perfectly correlated and the variances of their growths are equal. These relations generally do not hold in models with a stochastic opportunity set and certainly are not present in the economy. Incorporating a stochastic opportunity set into the dynamic programming problem typically requires numerical methods, and this is no different for consumption-wealth felicity.

[27] This is the marginal felicity of wealth holding consumption constant; the marginal felicity of wealth holding the consumption-wealth ratio, z, constant is always positive.

[28] Equation (35) is derived by substituting $G(t) = \mathrm{e}^{-\delta t} v'(z^*)$ and $V_z^{-1}(G,t) = z^*$ into (30).

One exception is a generalization of logarithmic felicity of consumption to log homogeneous felicity of consumption and wealth, $u(C, W, t) = \Delta(t)\ell n[W \cdot V(z, t)] = \Delta(t)\ell n W + F(z, t)$.[29] For log homogeneous felicity, the indirect utility of wealth is $J(W, \mathbf{x}, t) = G(t)\ell n W + g(\mathbf{x}, t)$, and consumption is a deterministic fraction of wealth even with a stochastic opportunity set. For example, an investor with $u(C, W, t) = e^{-\delta t}[\ell n W + f(z)]$ who lives until time T and has no motive for bequest has indirect utility with $G(t) = [1 - e^{-\delta(T-t)}]/\delta$ and optimal consumption of $C^*(t) = f'^{-1}([1-e^{-\delta(T-t)}]/\delta)W$. For the limiting case of the previously examined Cobb-Douglas felicity, with $f(z) = (1-\theta)\ell n\, z$, consumption is $(1-\theta)\delta[1-e^{-\delta(T-t)}]^{-1}$ which is in the constant proportion $1-\theta$ to the consumption of an investor with log felicity of consumption alone.

Log homogeneous felicity shares the feature of log utility of consumption that optimal consumption does not depend on the investment opportunities. Because of this, there is no need to hedge any changes in the opportunities, and consumption remains proportional to and perfectly correlated with wealth.

While numerical methods are required to solve the stochastic-environment investment problem for other consumption-wealth felicity functions, some insight about their characteristics can be determined from comparative statics. From (33), the two economy-wide variables that affect a homogeneous consumption-wealth felicity investor's propensity to consume are the interest rate and the market's Sharpe ratio. For Cobb-Douglas felicity, an increase in either one will decrease optimal consumption when relative risk aversion is greater than unity; that is, the cross-substitution effect exceeds the income effect when $\alpha < 0$. The opposite is true at lower risk aversions.

$$\frac{\partial z_{\text{CD}}}{\partial r} = \frac{-(1-\theta)\alpha}{1-\alpha(1-\theta)} \qquad \frac{\partial z_{\text{CD}}}{\partial S^2} = \frac{-(1-\theta)\alpha}{2(1-\alpha)[1-\alpha(1-\theta)]}. \tag{36}$$

Of course, the same is true for an investor with felicity of consumption which is the special case $\theta = 0$. The absolute magnitude of each effect is decreasing in θ.[30]

This result extends to all homogeneous consumption-wealth felicities; optimal consumption is increasing or decreasing in both the interest rate, r, and the market's Sharpe ratio, S, as α is positive or negative. Using the implicit function theorem[31]

$$\frac{\partial z^*}{\partial r} = -\frac{(1-\alpha)(v')^2}{\alpha v v''}\frac{\partial z_{\text{Merton}}}{\partial r} \qquad \frac{\partial z^*}{\partial S^2} = -\frac{(1-\alpha)(v')^2}{\alpha v v''}\frac{\partial z_{\text{Merton}}}{\partial S^2}. \tag{37}$$

verifying the claim since $\alpha < 1$, α and v must have the same sign, and $v'' < 0$.

[29] This is identical to a felicity function $\Delta(t)\ell n\, C + \Phi(z, t)$ with $\Phi(z, t) \equiv F(z, t) - \Delta(t)\ell n\, z$.

[30] Differentiating in (36), $\partial^2 z_{CD}/\partial r\,\partial\theta = \alpha[1-\alpha(1-\theta]^{-2}$. So $\partial z_{CD}/\partial r$ is negative (positive) and increasing (decreasing) in θ when α is positive (negative).

[31] Using (35), define the function $f(z; r, S^2) \equiv (v/v') - z - (1-\alpha)z_{\text{Merton}}/\alpha$. Then $\partial f/\partial z = -vv''/(v')^2$, and $\partial f/\partial r = -[(1-\alpha)/\alpha]\partial z_{\text{Merton}}/\partial r$. At the optimum, $f = 0$ so using the implicit function theorem, $\partial z/\partial r = -(\partial f/\partial r)/(\partial f/\partial z)$. A similar result holds for the Sharpe ratio.

V.3 Consumption-Wealth Felicity with Habit Formation

One interpretation of the role of wealth in the felicity function is similar to that of past consumption in models with habit formation. When a high level of consumption is expected to continue, a shortfall may lead to regret so the felicity of a given level of consumption will be decreasing in past consumption. Similarly currently high wealth and the anticipation of future high consumption might also have a depressing effect on current felicity. Of course, the anticipation of future consumption might instead afford a sustaining effect and provide high current felicity even when consumption is low.

Constantinides (1990) and Ingersoll (1992) examined a particular functional form of habit formation with a linear trade-off between current consumption and an exponentially smoothed average of past consumption. In this section we consider an extension of their models with felicity of the form $u(C, W, H, t) = e^{-\delta t}(C - aW - bH)^{\alpha}/\alpha$ with a linear trade-off among consumption, current wealth, and habit.[32] The measure of habit, H, is an exponentially smoothed average of past consumption, $H = \kappa \int e^{-\kappa s} C_{t-s} ds$.

Indirect utility depends on the habit as well as wealth, time, and any state variables. This felicity function, like those in the previous section, is homogeneous of degree α in C, W, and H. The derived utility function is $J(W, H, \mathbf{x}, t) = e^{-\delta t}G(\mathbf{x}, t)(W + \eta H)^{\alpha}/\alpha$. Since the evolution of $H, dH = \kappa(C - H)dt,$ is locally deterministic, habit affects the portfolio choice only through changing risk aversion; there is no portfolio for hedging against uncertain changes in H. Similarly, there is no specific portfolio for hedging against changes in anticipated future consumption which is measured here by wealth.

With a static opportunity set, the indirect utility function, J, is the solution to

$$0 = \tfrac{1}{2}\mathbf{w}'\boldsymbol{\Omega}\mathbf{w}W^2 J_{WW} + \big([r + \mathbf{w}'(\boldsymbol{\mu} - r\mathbf{1})]W - C\big) J_W + J_t + u(C, W, H, t) + \kappa(C - H)J_H \, . \qquad (38)$$

As shown in the Appendix, derived utility for this felicity function is

$$J(W, H, t) = e^{-\delta t}\left[z_{\text{Merton}} + \frac{\alpha a(1-\kappa\eta)}{1-\alpha} \right]^{\alpha-1} \frac{(W + \eta H)^{\alpha}}{\alpha(1-\kappa\eta)^{\alpha}}$$
$$\text{where} \quad \eta \equiv -\frac{r + \kappa - a - \kappa b - [(r + \kappa - a - \kappa b)^2 - 4ab\kappa]^{1/2}}{2a\kappa} \, . \qquad (39)$$

and z_{Merton} is the consumption rate of an investor with standard time-additive utility (the same α with $a = b = 0$). Optimal consumption is

$$C^* = \left(\frac{z_{\text{Merton}}}{1-\kappa\eta} + \frac{a}{1-\alpha} \right) W + \left[\left(\frac{z_{Merton}}{1-\kappa\eta} + \frac{\alpha a}{1-\alpha} \right)\eta + b \right] H \, . \qquad (40)$$

[32] The felicity function has been written so that for $a > 0$ high anticipated future consumption (i.e., wealth) creates regret and reduces current felicity just as does high past consumption. Both a and b can be negative in which case wealth and past consumption increase current felicity; however, the interior optimum for consumption can then be negative just as with HARA utility unless a nonnegativity constraint is explicitly imposed.

With habit added to the felicity function, consumption is no longer a constant fraction of wealth even with a constant opportunity set. The relative risk aversion of indirect utility is $(1-\alpha)(1+\eta H/W)^{-1}$ which also varies over time so the optimal leverage for an investor will change. Using wealth and habit in the felicity function permits a severance of the propensity to consume from the investor's risk aversion and should permit a great deal of flexibility in calibrating the model while maintaining stationarity.

VII Conclusions

This paper presents a new model of time-additive consumption-wealth utility. Like recursive utility, this model separates the roles of risk aversion and the intertemporal elasticity of consumption allowing it to be calibrated to a wider variety of data. Indeed, the observed equity premium and low interest rate — the joint puzzle of the equity premium puzzle — can be explained within this model without requiring a large relative risk aversion.

An advantage of this model is that dynamic programming can be used to determine optimal portfolio demands and consumption. A stochastic opportunity set can be accommodated easily in the model, though analytical solutions are not yet know except for the simplest situations as is the case for the Merton model. Nor can demands be easily aggregated to determine a representative agent unless all agents have very similar utility. Of course this is also true in the recursive utility formulation.

This model also provides an obvious way to interpret and analyze the difference between long- and short-run risks Fluctuations in consumption are short-lived risks while changes in wealth have long-term effects.

APPENDICES

A Simple Illustration of the Stochastic Differential Utility Consumption-Investment Problem

This appendix provides a simple derivation of the stochastic differential utility consumption-investment problem. In discrete time, the value function is given by the relation

$$v_t = \Gamma(c_t, \hat{v}_t) \quad \text{where} \quad \hat{v}_t \equiv \upsilon^{-1}(\mathbb{E}[\upsilon(\tilde{v}_{t+1}) \mid W_t, \mathbf{x}_t]). \tag{A1}$$

Γ is the aggregator which governs the tradeoff between consumption at time t and the currently assessed certainty equivalent, $\hat{v}_t$, of next period's utility. β is the subjective discount factor. The certainty equivalent is computed using a function υ which governs the evaluation of timeless risks conditional on the information available at time t as measured by the vector of state variables $\mathbf{x}_t$.

For Epstein-Zin preferences $\Gamma(C, \hat{v}) = [C^\rho + \beta \hat{v}^\rho]^{1/\rho}$ and $\upsilon(z) = z^\alpha$. In continuous-time, define the indirect utility certainty equivalent $\Psi(W, \mathbf{x}, t) = v_t$. Then

$$\Psi(W_t, \mathbf{x}_t, t) \equiv \lim_{\Delta t \to 0} \max_{c_t, \mathbf{w}_t} \{C_t^\rho \Delta t + e^{-\delta \Delta t} (\mathbb{E}[\Psi^\alpha(\tilde{W}_{t+\Delta t}, \tilde{\mathbf{x}}_{t+\Delta t}, t+\Delta t)])^{\rho/\alpha}\}^{1/\rho}. \tag{A2}$$

As in the discrete-time recursive model, consumption and the certainty equivalent of wealth will be proportional to wealth,

$$C = \Xi(\mathbf{x}, t) W \qquad \Psi(W, \mathbf{x}, t) = \psi(\mathbf{x}, t) W. \tag{A3}$$

Substituting (A6) into (A5) and using Ito's lemma on the second term

$$\begin{aligned} e^{-\delta dt}\left(\mathbb{E}[(\tilde{W}_{t+dt}\tilde{\psi}_{t+dt})^\alpha]\right)^{\rho/\alpha} &= W^\rho \psi^\rho \{1 - \delta dt + \rho \mathbb{E}[dW/W] + \tfrac{1}{2}\rho(\alpha-1)\operatorname{var}[dW/W] \\ &\quad + \rho \mathbb{E}[d\psi/\psi] + \tfrac{1}{2}\rho(\alpha-1)\operatorname{var}[d\psi/\psi] + \rho\alpha \operatorname{cov}[dW/W, d\psi/\psi]\} \\ &= W^\rho \psi^\rho \{1 - \delta dt + \rho(r + \mathbf{w}'(\mu - r\mathbf{1}) - \Xi)dt + \tfrac{1}{2}\rho(\alpha-1)\mathbf{w}'\mathbf{\Omega}\mathbf{w}dt \\ &\quad + \rho\, \mathbb{E}[d\psi/\psi] + \tfrac{1}{2}\rho(\alpha-1)\operatorname{var}[d\psi/\psi] + \rho\alpha \mathbf{w}'\mathbf{\Lambda}\psi_{\mathbf{x}}/\psi^{-1}dt\}. \end{aligned} \tag{A4}$$

where W and ψ are all evaluated at time t and $\psi_{\mathbf{x}} \equiv \partial\psi/\partial\mathbf{x}$ is a vector of the partial derivatives. Substituting (A6) and (A7) into (A5) and dividing through by ψW

$$\Psi(W_t, \mathbf{x}_t, t) \equiv \lim_{\Delta t \to 0} \max_{c_t, \mathbf{w}_t} \{C_t^\rho \Delta t + e^{-\delta \Delta t} (\mathbb{E}[\Psi^\alpha(\tilde{W}_{t+\Delta t}, \tilde{\mathbf{x}}_{t+\Delta t}, t+\Delta t)])^{\rho/\alpha}\}^{1/\rho}. \tag{A5}$$

The optimal consumption and portfolio weights can now be determined as solutions to the sub-problems embedded in (A8). The optimal portfolio is

$$C = \Xi(\mathbf{x}, t) W \qquad \Psi(W, \mathbf{x}, t) = \psi(\mathbf{x}, t) W. \tag{A6}$$

Substituting (A6) into (A5) and using Ito's lemma on the second term

$$
\begin{aligned}
e^{-\delta dt}\left(\mathbb{E}[(\tilde{W}_{t+dt}\tilde{\psi}_{t+dt})^{\alpha}]\right)^{\rho/\alpha} &= W^{\rho}\psi^{\rho}\left\{1-\delta dt+\rho\mathbb{E}[dW/W]+\tfrac{1}{2}\rho(\alpha-1)\operatorname{var}[dW/W]\right. \\
&\quad \left.+\rho\mathbb{E}[d\psi/\psi]+\tfrac{1}{2}\rho(\alpha-1)\operatorname{var}[d\psi/\psi]+\rho\alpha\operatorname{cov}[dW/W,d\psi/\psi]\right\} \\
&= W^{\rho}\psi^{\rho}\left\{1-\delta dt+\rho(r+\mathbf{w}'(\mu-r\mathbf{1})-\Xi)dt+\tfrac{1}{2}\rho(\alpha-1)\mathbf{w}'\boldsymbol{\Omega}\mathbf{w}dt\right. \\
&\quad \left.+\rho\,\mathbb{E}[d\psi/\psi]+\tfrac{1}{2}\rho(\alpha-1)\operatorname{var}[d\psi/\psi]+\rho\alpha\mathbf{w}'\boldsymbol{\Lambda}\psi_{\mathbf{x}}/\psi^{-1}dt\right\}.
\end{aligned}
\tag{A7}
$$

Substituting (A10) and (A6) and (A7) into (A5) and dividing through by ψW

$$
\begin{aligned}
0=\max_{\Xi,\mathbf{w}}\Big[&\rho^{-1}(\Xi^{\rho}\psi^{-\rho}-\delta)+r+\mathbf{w}'(\boldsymbol{\mu}-r\mathbf{1}-\Xi+\tfrac{1}{2}(\alpha-1)\mathbf{w}'\boldsymbol{\Omega}\mathbf{w} \\
&+\mathbb{E}[d\psi/\psi]/dt+\tfrac{1}{2}(\alpha-1)\operatorname{var}[d\psi/\psi]/dt+\alpha\psi^{-1}\mathbf{w}'\boldsymbol{\Lambda}\psi_{\mathbf{x}}\Big].
\end{aligned}
\tag{A8}
$$

The optimal consumption and portfolio weights can now be determined as solutions to the sub-problems embedded in (A8). The optimal portfolio is

$$
\mathbf{w}^*=\frac{1}{1-\alpha}\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu}-r\mathbf{1})+\frac{\alpha}{(1-\alpha)\psi}\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\psi_{\mathbf{x}} \tag{A9}
$$

just as in the standard model. Optimal consumption is the solution to

$$
\max_{\Theta}\ \rho^{-1}\Xi^{\rho}\psi^{-\rho}-\Xi \quad\Rightarrow\quad \Xi^*(\mathbf{x},t)=\psi^{\rho/(\rho-1)}(\mathbf{x},t). \tag{A10}
$$

Substituting (A10) and (A9) into (A8) allows us to determine ψ from the exogenous variables.

$$
\begin{aligned}
0=&\frac{1-\rho}{\rho}\psi^{\rho/(\rho-1)}-\frac{\delta}{\rho}+r+\frac{(\boldsymbol{\mu}-r\mathbf{1})'\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu}-r\mathbf{1})}{2(1-\alpha)}+\frac{\alpha\psi_{\mathbf{x}}'\boldsymbol{\Lambda}'\boldsymbol{\Omega}^{-1}(\boldsymbol{\mu}-r\mathbf{1})}{2(1-\alpha)} \\
&+\frac{\mathbb{E}[d\psi/\psi]}{dt}+\frac{\alpha-1}{2}\frac{\operatorname{var}[d\psi/\psi]}{dt}+\frac{\alpha^2}{(1-\alpha)\psi}\psi_{\mathbf{x}}'\boldsymbol{\Lambda}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\psi_{\mathbf{x}}.
\end{aligned}
\tag{A11}
$$

Solution of the Consumption-Wealth Portfolio Problem with Habit

For the felicity function $e^{-\delta t}(C-aW-bH)^{\alpha}/\alpha$, take a trial solution for derived utility of $J(W,H,t)=e^{-\delta t}A^{\alpha-1}(W+\eta H)^{\alpha}/\alpha$. Substituting into (38) and simplifying leaves

$$
0=(W+\eta H)\left[\frac{S^2}{2(1-\alpha)}-\frac{\delta}{\alpha}+A(1-\kappa\eta)^{\alpha/(\alpha-1)}\frac{1-\alpha}{\alpha}\right]+rW-\kappa\eta H-(aW+bH)(1-\kappa\eta). \tag{A12}
$$

The terms in W and H must each be zero so

$$
\begin{aligned}
0 &= \frac{S^2}{2(1-\alpha)} - \frac{\delta}{\alpha} + A(1-\kappa\eta)^{\alpha/(\alpha-1)}\frac{1-\alpha}{\alpha} + r - a(1-\kappa\eta) \\
0 &= \eta\left[\frac{S^2}{2(1-\alpha)} - \frac{\delta}{\alpha} + A(1-\kappa\eta)^{\alpha/(\alpha-1)}\frac{1-\alpha}{\alpha} - \kappa\right] - b(1-\kappa\eta)\,.
\end{aligned} \tag{A13}
$$

Multiply the first by η and subtract the second to leave a quadratic equation in η

$$
0 = \eta(r+\kappa) - \eta a(1-\kappa\eta) + b(1-\kappa\eta) \tag{A14}
$$

with solutions

$$
\eta \equiv -\frac{r+\kappa-a-\kappa b \pm [(r+\kappa-a-\kappa b)^2 - 4ab\kappa]^{1/2}}{2a\kappa}\,. \tag{A15}
$$

as given in (40). The negative root must apply since when habit should not affect direct utility (b = 0), it cannot affect indirect utility (η = 0). The first line of (A13) can now be solved for A

$$
\begin{aligned}
A &= (1-\kappa\eta)^{-\alpha/(\alpha-1)}\frac{\alpha}{1-\alpha}\left[\frac{\delta}{\alpha} - r - \frac{S^2}{2(1-\alpha)} + a(1-\kappa\eta)\right] \\
&= (1-\kappa\eta)^{-\alpha/(\alpha-1)}\left[z_{\text{Merton}} + \frac{\alpha a(1-\kappa\eta)}{1-\alpha}\right].
\end{aligned} \tag{A16}
$$

From the envelope condition $u_C = J_W - \kappa J_H$, so optimal consumption is

$$
\begin{aligned}
C^* &= A(W+\eta H)(1-\kappa\eta) + aW + bH \\
&= (1-\kappa\eta)^{-1}\left[z_{\text{Merton}} + \frac{\alpha a(1-\kappa\eta)}{1-\alpha}\right](W+\eta H) + aW + bH \\
&= \left(\frac{z_{\text{Merton}}}{1-k\eta} + \frac{a}{1-\alpha}\right)W + \left[\left(\frac{z_{Merton}}{1-k\eta} + \frac{\alpha a}{1-\alpha}\right)\eta + b\right]H
\end{aligned} \tag{A17}
$$

as stated in (40) in the text.

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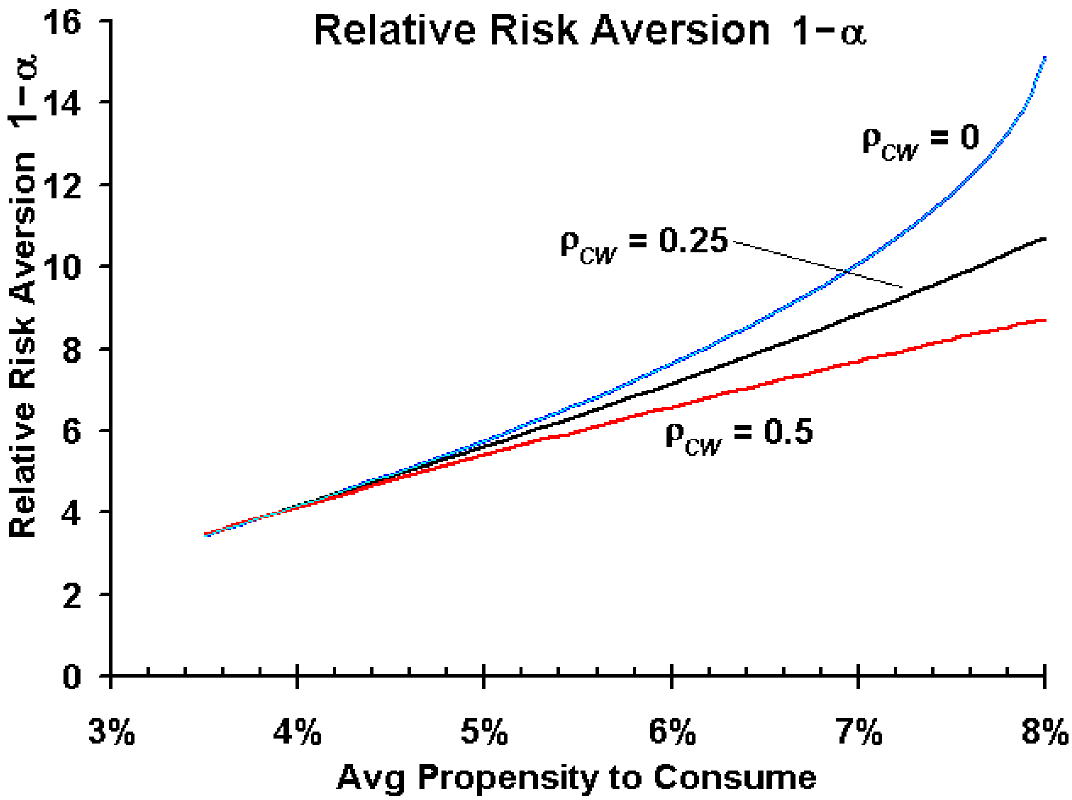

Figure 1: Relative Risk Aversion
This figure plots the constant relative risk aversion, $1-\alpha$, that is consistent with the Mehra Prescott data used in describing the Equity Premium Puzzle as a function of the average propensity to consume. The felicity function is Cobb-Douglas $u(C, W, t) = e^{-\delta t}C^{\alpha(1-\theta)}W^{\alpha\theta}/\alpha$. The Mehra-Prescott data is equity expected rate of return and standard deviation (μ_W = 6.98%, σ_W = 16.54%), consumption growth rate and standard deviation (ζ_C = 1.83%, σ_C = 3.57%), and real interest rate (r = 0.80%). The correlation between wealth and consumption is 0.22, and the subjective discount rate is of δ = 1%. In a steady state the average growth rates for wealth and consumption must be equal so $\mu_W + C/W = \zeta_C$ implying C/W = 5.15% and a relative risk aversion of $1-\alpha = 5.87$.

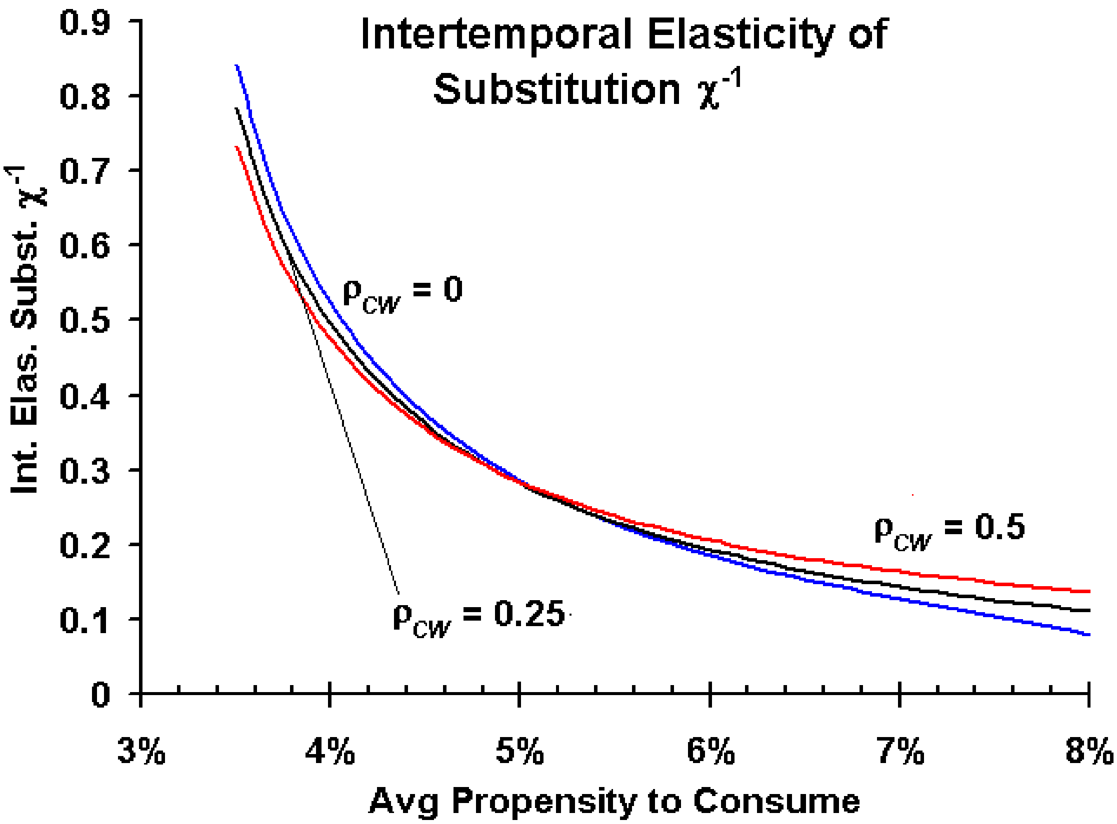

Figure 2: Intertemporal Elasticity of Substitution of Consumption χ^{-1}

This figure plots the elasticity of the intertemporal substitution of consumption, χ^{-1}, that is consistent with the Mehra Prescott data used in describing the Equity Premium Puzzle as a function of the average propensity to consume. The felicity function is Cobb-Douglas $u(C, W, t) = e^{-\delta t}C^{\alpha(1-\theta)}W^{\alpha\theta}/\alpha$. The Mehra-Prescott data is equity expected rate of return and standard deviation ($\mu_W = 6.98\%$, $\sigma_W = 16.54\%$), consumption growth rate and standard deviation ($\zeta_C = 1.83\%$, $\sigma_C = 3.57\%$), and real interest rate ($r = 0.80\%$). The correlation between wealth and consumption is 0.22, and the subjective discount rate is of $\delta = 1\%$. In a steady state the average growth rates for wealth and consumption must be equal so $\mu_W + C/W = \zeta_C$ implying $C/W = 5.15\%$ and an EISC of $\chi^{-1} = 0.264$.

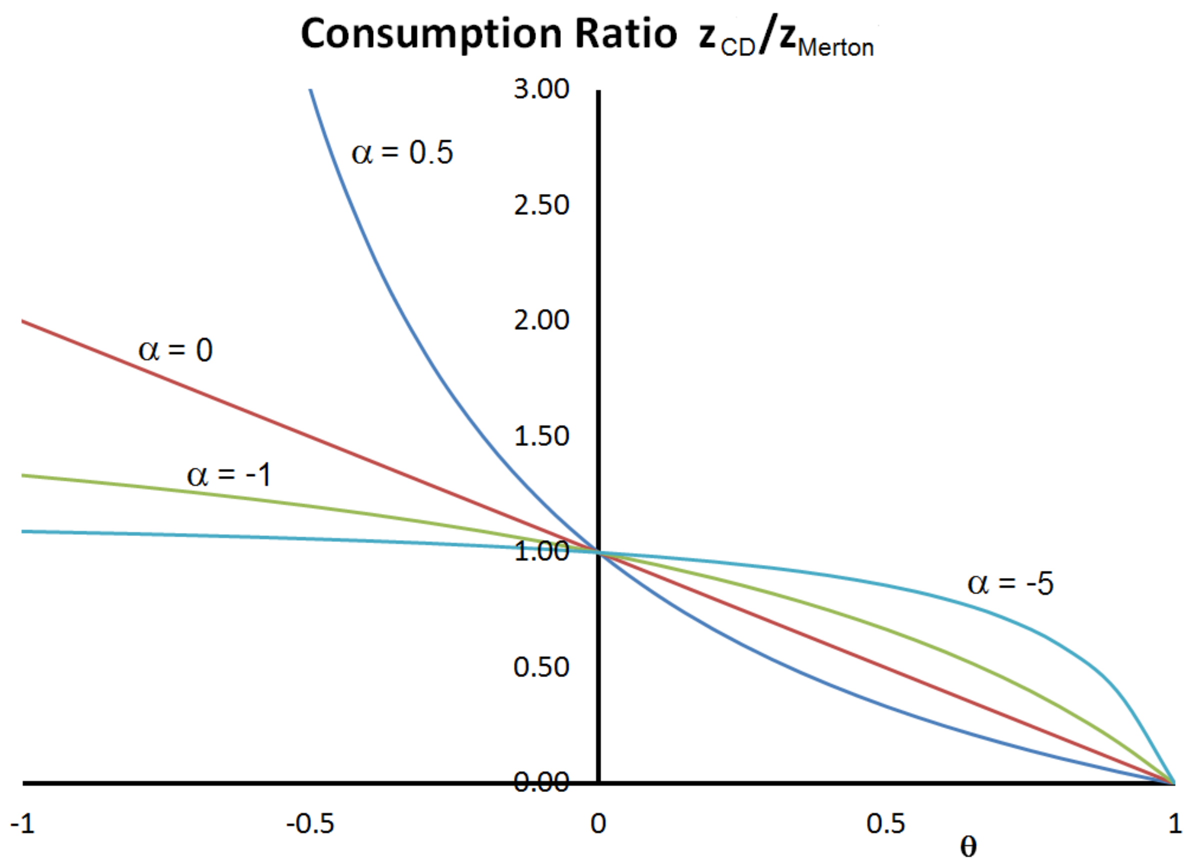

Figure 3: Comparison of Propensity to Consume in Consumption-Wealth Felicity and Standard Models

This figure plots the ratio of the propensity to consume in the Cobb-Douglas consumption-wealth felicity model to that in the standard model. The felicity function is $u(C, W, t) = e^{-\delta t}C^{\alpha(1-\theta)}W^{\alpha\theta}/\alpha$. The propensities to consume are $z_{\text{Merton}} = [\delta - \alpha r - \alpha S^2/2(1-\alpha)]/(1-\alpha)$ in the standard model and $z_{CD}/z_{\text{Merton}} = (1-\theta)(1-\alpha)/[1-\alpha(1-\theta)]$ in the consumption-wealth felicity model.