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Harry Mamaysky
Yale School of Management

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Harry Mamaysky*

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Comments Welcome

*Yale School of Management, 135 Prospect St., Box 208200, New Haven, CT, 06520-8200. Email: harry.mamaysky@yale.edu. Phone: (203) 436-0649. Fax: (203) 436-0630. I would like to thank Ray Fair, Hua He, Jonathan Ingersoll, Andrew Jeffrey, Leonid Kogan, Geert Rouwenhorst, Matthew Spiegel, Raman Uppal, Jiang Wang, and seminar participants at Yale and the 2001 European Finance Association meeting for very helpful comments and discussions.

Interest Rates and the Durability of Consumption Goods

Abstract

In this article I study an economy with irreversible durable investment and investors who consume a durable and a nondurable good. In a general equilibrium setting, these assumptions lead to endogenous variation in the implied risk aversion of investors and in the term structure of interest rates. In the model, the magnitude of the intertemporal elasticity of substitution places certain restrictions on the joint dynamical behavior of durable consumption, nondurable consumption, and the yield curve. Tests of the model using postwar U.S. data are supportive of these restrictions. However, while the model is able to generate a relatively large term spread, the level and the variation of the resultant short rate are not empirically plausible. An approximate closed form solution of the model is derived.

JEL Classification: G1, D5, E2

Contents

1 Introduction

The relationship between interest rates and consumption is one of the central issues in financial economics. While it is intuitively clear that a relationship between these quantities should exist, the exact nature of this relationship, as well as the economic forces which bring it about, are far less obvious. However, the potential payoffs from having a good understanding of how interest rates and consumption interact are vast. For example, such knowledge would render the predictability present in consumption data useful for forecasting interest rate movements, and vice versa. Alternatively, variation in the joint behavior of interest rates and consumption may shed light on how investor preferences differ over time and across countries.

Unfortunately relatively little is know about the joint behavior of interest rates and consumption. Traditional models of this interaction are based on the following simple observation: If people choose consumption in an optimizing way, then consumption growth should be related to available investment opportunities, and therefore to interest rates. In models with a single, nondurable consumption good and time separable preferences, such as Lucas (1978) and Breeden (1979), this relationship takes on the following form

$$\text{Consumption Growth} = \text{Contant} + \text{IES} \times r. \tag{1}$$

The intertemporal elasticity of substitution (IES) is a measure of the willingness of consumers to postpone consumption in response to changes in their investment opportunities. As interest rates rise, and hence as investment opportunities look more favorable, consumers will postpone consumption in favor of investment.

Under the assumption that consumption consists of only nondurables and services, empirical research has shown that the model in (1) is a poor description of aggregate consumption and interest rate data (see Hansen and Singleton (1983), Hall (1988), Hansen and Jaganathan (1993) and Mankiw and Zeldes (1996)). An obvious argument against the validity of the relationship in (1) is that nondurables and services may not be the only relevant consumption goods. Cognizant of this objection, past researchers have pointed out that the restriction in (1) may still hold even if other consumption goods exist, as long as consumers have separable preferences: $u(c_1, \ldots, c_N) = u_1(c_1) + \cdots + u_N(c_N)$ (see Hansen and Singleton (1983)).

Unfortunately, the claim that utility over consumption goods is separable has little empirical support. Dunn and Singleton (1986) and Ogaki and Reinhart (1998) show, for example, that the representative consumer has a nonseparable utility function over consumption of nondurables and services, on the one hand, and durables on the other (see also Mankiw (1985)). Furthermore, these papers suggest that models which take into account consumer preferences over durable goods are better able to account for the joint behavior of consumption (durable and nondurable) and asset returns (see also Heaton (1993,1995)).[1] Durability, therefore, seems to be an important ingredient in any recipe which purports to explain how and why interest rates are connected to consumption.

[1] In the remainder of the paper, nondurable consumption will refer to nondurables and services, and durable consumption will refer to durables and investment into residential real estate. The classifications are taken from the U.S. Department of Commerce.

The approach taken by Dunn and Singleton (1986) and by Ogaki and Reinhart (1998) is to estimate moment conditions implied by the optimal consumption choice of investors with nonseparable utilities over nondurable and durable goods. Indeed this approach, of looking at restrictions implied by first order conditions, has become a mainstay of empirical consumption asset pricing. Because it does not require solving for optimal investment policies of investors, it is amenable to empirical work. However papers which look only at first order conditions have the following drawback: Because they do not actually solve for the equilibrium in their model economies, they are unable to relate consumption and interest rates to economic fundamentals, such as the stocks of nondurable and durable capital. After all, how many houses have already been built to serve a given population must have an important affect of the latter's consumption and investment decisions. Any complete theory of the joint behavior of interest rates and consumption must take into account the fact that these quantities are outcomes of investor decisions, and must therefore be related to the amounts of nondurable and durable capital to which these investors already have access.

In the present paper, I formulate and solve a general equilibrium model in which investors consume durable and nondurable goods. Rather than simply characterize optimal consumption policies, I am able to solve for the optimal consumption and investment policies of investors. At the same time, I solve for the endogenous market clearing rates of interest in the economy. Furthermore, I am able to relate all three quantities, namely consumption, investment, and interest rates, to economic primitives, which in the model are the nondurable and durable capital stocks. The payoff of this approach is twofold. First, having a self consistent equilibrium model of the economy allows us to gain insights into the economic forces which connect interest rates with consumption. Second, the model yields a rich set of testable empirical implications.

For example, Figure 1 shows a time-series of the correlation (computed in rolling windows) between the shares of GNP devoted to durable and nondurable consumption, and between the share of GNP devoted to durable consumption and the term spread.[2] Even a casual inspection of the graph suggests the two correlations are intimately related. The model in this paper suggests that this relationship is not an artifact of the data, but instead arises from a very fundamental economic interaction (discussed shortly). As this paper will argue, whether the shares of GNP devoted respectively to nondurable and durable consumption tend to be high at the same time or not, determines how the term spread and expected excess stock returns fluctuate over the business cycle.[3] Furthermore, this model provides a theoretical justification for why ratios such as nondurable consumption to aggregate wealth have predictive power for asset returns (see Lettau and Ludvigson (2001)).

The model in this paper is a modification of Cox, Ingersoll, Ross (1985a). Agents in the model

[2]In this paper, the term spread is the difference in yields between a portfolio of intermediate maturity (approximately 5 years) U.S. government bonds and 3 month t-bills.

[3]Previous research, for example Fama and French (1989), Chen (1991), and Stock and Watson (1998), found that the term spread is countercyclical, as are excess expected stock returns. The theoretical analysis of this paper suggests that this does not necessarily have to be the case, and the empirical analysis of this paper suggests that this countercyclical behavior is true usually, but not always.

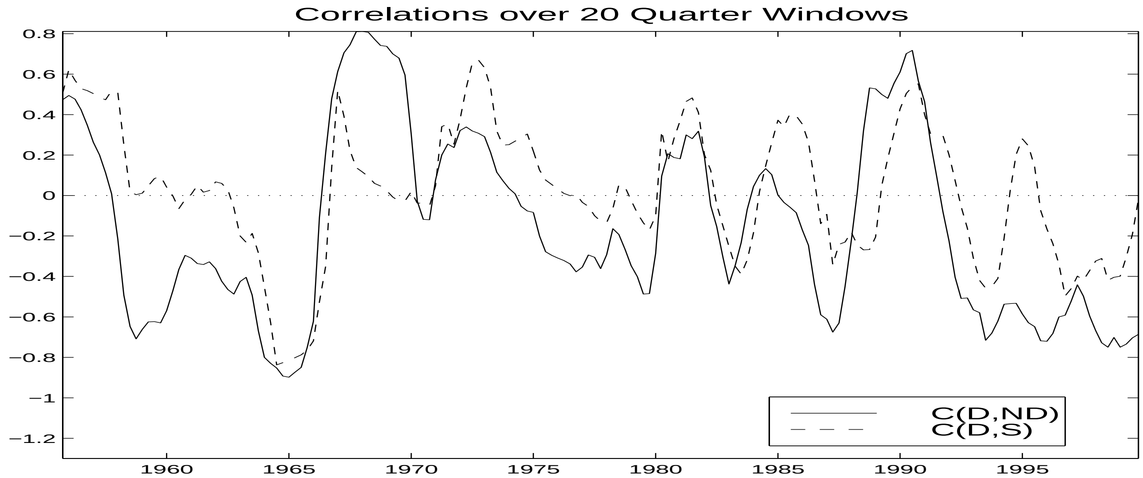

Figure 1: The solid line shows the correlation between the ratio of durables over GNP and the lagged ratio of nondurables over GNP (all variables are in real terms). The dashed line shows the correlation between durables over GNP and the term spead (the difference between 5 year government yields and the t-bill yield). Correlations are computed in rolling 20 quarter windows.

are assumed to have preferences over consumption given by

$$u(c,z) = \frac{\left(c^{\delta} z^{1-\delta}\right)^{1-\gamma}}{1-\gamma}$$

where c is the instantaneous nondurable good consumption, and z is the service flow derived from an agent's holdings of a durable good. Agents are able to invest their holdings of the nondurable good in a production technology with a constant investment opportunity set. Furthermore, at any time, they are able to transfer a unit of nondurable into a unit of the durable good. In order to qualitatively capture aggregate investment behavior, I assume that such transfers are irreversible. Once a unit of nondurable has been transferred into a unit of the durable good, the reverse transfer is technologically infeasible. I solve the model numerically, and also obtain an approximate closed form solution using the method of perturbation analysis.

I show that preferences over durable and nondurable goods, and the fact that durable investment is irreversible, are the driving forces of the results in the paper. Unless both assumptions are made, interest rates in this model are constant.[4] Since investment into durables is irreversible, agents choose to invest only after periods of high growth in the nondurable stock. Hence durable investment is procyclical. The key feature of equilibrium in the model is that agents optimally adjust their consumption of nondurables as they approach the investment point for durables. In

[4]It is well known that in the Cox, Ingersoll, and Ross (1985a,b) representative agent economy, if the production opportunity set is constant, the short term interest rate turns out to be constant as well, implying that the term structure of interest rates is flat. A nontrivial term structure can be obtained in this setting by assuming exogenous variation in the production opportunity set. Two prominent models of the term structure which do not need to assume exogenous state variables are Dumas (1989) and Wang (1996). Both models look at an economy with two groups of agents that have different risk aversions inside the HARA class. This paper shows that endogenous variation in interest rates can be obtained by having heterogeneous consumption goods, rather than heterogeneous investors.

particular, the ratio of nondurable consumption to the total nondurable capital stock (c/K) can increase or decrease as the durable investment point draws near. It will decrease if the intertemporal elasticity of substitution (or $1/\gamma$) is sufficiently high. Intuitively, if people are sufficiently willing to substitute consumption through time they will defer going out to dinner when they are close to buying a house. If not, c/K will increase as durable investment becomes imminent.

Consider the case where the c/K ratio is decreasing in times of high durable investment. First we note that due to irreversibility, this implies that c/K is decreasing in K. Therefore a 1% increase in the nondurable stock K will be accompanied by a less than 1% increase in consumption c. The elasticity of consumption with respect to capital will therefore be low around times of high durable investment. In this paper, I show that the implied risk aversion of investors is directly proportional to this elasticity. The intuition is quite clear: Since agents dislike uncertainty only to the extent that it affects their actual consumption, and since consumption becomes less sensitive to production shocks, agents are willing to bear higher amounts of production risk, and are therefore less risk averse.

The short term interest rate which induces zero borrowing and lending therefore rises in times when durable investment is high. Since the term structure reflects expectations of future short rates, it becomes downward sloping. Hence if the IES is sufficiently high, the ratio of nondurable consumption to capital (c/K) and durable investment are negatively correlated. Furthermore, durable investment should be negatively correlated with the term spread. Since implied risk aversion is lowest when new investment is about to occur (after periods of high capital growth) the expected excess returns on equity should be lowest at this point, and therefore countercyclical.

Of course all of these conclusions are reversed if the IES is low, and therefore if c/K increases when durable investment is high. And it is this observation which provides the main empirical predictions of the model. One of these is that if the shares of GNP devoted to nondurable and durable consumption are negatively correlated, hence suggesting that the IES is high, then the term spread should be countercyclical. Figure 1 is supportive of this prediction. More formal statistical tests are carried out later in the paper, and reach the same conclusion. Furthermore, the above argument suggests that the ratio of nondurable consumption to the capital stock should be positively correlated with the term spread regardless of the IES (simply repeat the arguments of the preceding paragraphs for the case when c/K increases when durable investment is high). This too is supported by the data. On a cautionary note, the time variation in the above correlations suggests that certain parameters which the model assumes to be constant are indeed fluctuating (perhaps slowly) over time.

Despite this empirical success, the model is unable to match certain salient moments of the data. For a reasonable choice of parameters, the level of the short rate generated by the model is too high. Furthermore the variation in the short rate, as a function of the capital stocks in the economy, is too low. On the other hand, the curvature of the term structure (relative to the highest and lowest possible short rates) is sufficiently high to be empirically plausible. The underlying cause of these shortcomings is the model's inability to separate out local risk aversion from the intertemporal elasticity of substitution. A higher elasticity implies more short rate variation, yet further raises the already implausible level of the short rate. Given the empirical success of the model's predictions, as well as this inability of the model to match certain moments of the data, it would seem that the "correct" asset pricing model should take into account irreversible durable and nondurable consumption, as well as a separation between local risk aversion and the intertemporal elasticity of substitution.

The rest of the paper proceeds as follows. Section 2 formulates the model. Section 3 shows how the equilibrium asset prices in this economy are derived. Section 4 analyzes the some base cases of the model, and provides an upper and lower bound for the value function. Section 5 analyzes the general case of the present model. Section 6 presents the empirical results. Section 7 concludes. All proofs and the perturbation analysis are in the Appendix.

2 The Model

The underlying uncertainty of the economy is characterized by a 1 dimensional standard Brownian motion $B = \{B_t : t \geq 0\}$ defined on its filtered probability space $(\Omega, \mathcal{F}, F, \mathcal{P})$. The filtration $\mathbf{F} = \{\mathcal{F}_t : t \geq 0\}$ represents the information revealed by B over time.

The economy contains a nondurable good (the productive capital) and a durable good. The nondurable good acts as the numeraire. There is a production technology for the nondurable in the economy. This technology transforms units of nondurable today into units of nondurable tomorrow. Units of nondurable may also be transformed into units of the durable good. However, units of durable good may not be transformed back into the nondurable. Hence investment in the durable good is irreversible. The stock of durable good depreciates over time to reflect the effects of physical deterioration. The aggregate nondurable stock K_t and the aggregate durable stock z_t evolve according to

$$dK(t) = \mu K dt + \sigma K dB(t) - c(t)dt - d\Phi(t) \tag{2}$$

$$dz(t) = -\theta z(t)dt + d\Phi(t) \tag{3}$$

Here $c(t)$ is aggregate nondurable consumption, $d\Phi(t)$ is the time t aggregate investment into the durable, and θ is the durable depreciation rate. Here μ and σ are constants which characterize the production opportunities available for the nondurable goode. We require that $c(t) \geq 0$, and note that irreversibility of investment into the durable implies that $d\Phi(t) \geq 0$.

In (2–3), the stochastic process $\Phi(t)$ is the cumulative amount of nondurable transfered into the durable good as of time t. Investment can occur either in infinitesimal flows or in lumps.[5] As will

[5]More formally (following Hindy and Huang (1993)) let us define $\mathbf{X}^+$ as the space of all processes x with paths that are positive, increasing, and right continuous. An increasing function $x(\cdot)$ has a finite left limit at any t, denoted by $x(t^-)$. The convention used in this paper is that $x(0^-) = 0$. A jump of $x(\cdot)$ at τ is denoted by $\Delta x(\tau) = x(\tau) - x(\tau^-)$. We will assume that $\Phi(t) \in \mathbf{X}^+$. For any $\omega \in \Omega$ the points of discontinuity of $\Phi(\omega, t)$ correspond to the times when agents transfer a non-infinitesimal amount of nondurable into the durable good. The transfer process $\Phi(t)$ has an absolutely continuous component over those times when agents transfer nondurable into the durable at a rate of $d\Phi(\omega, t)/dt$ per unit time. Furthermore, $\Phi(\omega, t)$ may have a singular component.

The context in which the singular process arises in the present setting is the following. Let $B(t)$ be a standard Brownian motion on $\mathbf{F}$, and let c be some constant. Define a process $A(t) \equiv \sup_{s \leq t}(B_s - c)^+$. The process $A(t)$ will be singularly continuous (as opposed to absolutely continuous) and will be referred to as a *singular* process. Notice that become clear, for the optimal policy, infinitesimal investment occurs when it is necessary to keep the pair $\{K(t), z(t)\}$ inside a region of no-investment. An infinitesimal amount of nondurable will be transferred into the durable when the pair reaches the boundary of the region, and the transfer is only large enough so as to push the pair back into the region's interior. Because the state variables in the model evolve in a continuous way, once the pair $\{K(t), z(t)\}$ is either on the boundary or inside of the no-investment region, it will remain inside this region. Lumpy investment can occur if at time 0 the pair $\{K_0, z_0\}$ lies outside of the no-investment region. The lumpy control is then exerted to bring the pair to the boundary of the region. In what follows we assume, without loss of generality, that $\{K_0, z_0\}$ is inside the no-investment region.

Each unit of the durable good produces a consumption stream which is valuable to the agents in the model. Furthermore agents may consume out of their own nondurable stock. Agents' time t utility over consumption from the durable and nondurable good is given by

$$u(c,z) = \frac{\left(c^{\delta} z^{1-\delta}\right)^{1-\gamma} - 1}{1-\gamma}, \tag{4}$$

where c is the consumption rate from the nondurable, z is the service flow from z units of the durable good, $\gamma \geq 0$, and $\delta \in (0,1)$. The case of $\gamma = 1$ corresponds to separable preferences given by $u = \delta \log c + (1-\delta) \log z$. Going forward, it will be convenient to drop the -1 and to rewrite the above utility function as

$$u(c,z) = \frac{c^A z^B}{A+B}. \tag{5}$$

Here $A = \delta(1-\gamma)$ and $B = (1-\delta)(1-\gamma)$. Given the above restrictions on δ and γ, we have that (1) $A > 0, B > 0$ and $A + B < 1$, or (2) $A < 0, B < 0$. The case of $A + B = 0$ corresponds to log separable utility.

Agents maximize a utility of the form

$$\mathrm{E}_0 \left[\int_0^{\infty} e^{-\rho t} u(c(t), z(t)) dt \right] \tag{6}$$

by choosing consumption and investment processes $\{c, \Phi\}$, subject to the regularity conditions given in Lemma 4 in the Appendix, and capital stock dynamics given by (2,3). The economy contains a continuum of competitive investors, all of whom have identical preferences given by (6).

We make one additional assumption: all agents begin life with identical ratios of capital stocks, or $K_i(0)/z_i(0)$ is the same across all individuals. Given this assumption and the fact that all agents have identical preferences and beliefs, the economy can be characterized by the behavior of a single representative agent whose initial capital stocks are $K(0) = \sum_i K_i(0)$, and $z(0) = \sum_i z_i(0)$. It is then easy to show $K(0)/z(0)$ equals $K_i(0)/z_i(0)$ for all i. Rubinstein (1974) contains a discussion of these types of aggregation results in a setting with a single consumption good. Going forward, therefore, we will only be concerned with the problem of the aggregate agent.

(i) $A(t)$ is non-decreasing and therefore of finite variation, and (ii) the controlled Brownian motion $B^*(t) \equiv B(t) - A(t)$ will always be less than or equal to c.

2.1 Discussion of the Model

This model can be thought of as an extension of a simplified version of the Cox, Ingersoll, and Ross (1985a) model. In order to generate a non-trivial term structure in their model, the authors needed to assume an exogenous state variable. The obvious drawback of this approach is that an exogenously assumed source of uncertainty is difficult to interpret.

Another related model is Hindy and Huang (1993). They assume that agents only have utility over durable consumption. Alternatively, one can interpret the preference structure in their case as allowing for local substitution of nondurable consumption. As in Cox, Ingersoll, and Ross (1985a), the Hindy and Huang model with a constant investment opportunity set will have constant implied risk aversion, and therefore, constant interest rates.

The constant investment opportunity set CIR model, and the Hindy and Huang model represent two base cases of the present model corresponding, respectively, to $B = 0$ and $A = 0$.

Other related papers include Detemple and Giannikos (1996), who consider investors who consume nondurable and durable goods. In their model agents derive "status" from their durable goods purchases. This makes the irreversibility constraint faced by these investors non-binding. Grossman and Laroque (1990) and Cuoco and Liu (2000) study the control problem of an investor who can make costly transfers of productive capital to and from a durable good. In a setting similar to the one in this paper, Damgaard, Fuglsbjerg, and Munk (2000) study the control problem of an investor who consumes durables and nondurables. However they do not consider the asset pricing implications of durable and nondurable consumption. Hindy, Huang, and Zhu (1997) model an economy with investors who consume a durable good and form a habit over past durable good consumption. Dumas (1992) analyzes the exchange rate between two otherwise identical nondurable goods in spatially separated economies. The paper is similar to this one in that frictions exist for transfers between the capital stocks. One major difference is that both stocks in Dumas (1992) are of a nondurable consumption good.

There is also an extensive and related literature on the investment decisions of firms. Kogan (2000) studies the stock prices of profit maximizing firms which invest in a durable good in an economy where investors' preferences over durable and nondurable consumption are separable. Other related papers include Abel and Eberly (1996), Bertola and Caballero (1994), and Dixit (1991). There has also been a considerable amount of work in macroeconomics on the interaction of investment decisions and the business cycle. Rouwenhorst (1995) provides a discussion of how this literature is relevant to topics in asset pricing.

3 The Equilibrium

In this section we will analyze the solution to the representative agent's control problem, as well as the asset prices which obtain in equilibrium.

3.1 An Agent's Control Problem

The solution of a single agent's problem will give us the solution to the equilibrium of the economy as long as (1) all agents have identical preferences, and (2) all agents start at time 0 with identical ratios of nondurable to durable. We will assume that both of these conditions hold.

The nondurable consumption in the current problem occurs at a rate $c(t)$. Investment into the durable good, however, may have a singular component. The latter problem has been studied in the literature as well. See papers by Hindy and Huang (1993), Dumas (1991), Shreve and Soner (1994) and the book by Harrison (1990), among others. The joint control of $c(t)$ and the durable investment process $\Phi(t)$ can be handled in much the same way as the control of $\Phi(t)$ alone. It turns out sufficient conditions for this problem are similar to the Bellman conditions of dynamic programming. This section will heuristically discuss the optimality conditions for the value function and control. The section concludes with the statement of a verification theorem. A more rigorous discussion, as well as the proof of the verification theorem, are provided in the Appendix.

Let $J(K(t), z(t), t)$ be the value function of a single agent. The solution of the control problem in (6) satisfies the following Bellman type equation

$$\max\left[\sup_{c}\left\{e^{-\rho t}\frac{c^A z^B}{A+B} + J_t - \theta z J_z + (K\mu - c(t))J_K + \tfrac{1}{2}K^2\sigma^2 J_{KK}\right\}, J_z - J_K\right] = 0 \qquad (7)$$

where J_z and J_K are partial derivatives. We interpret this inequality as follows. The state space of the problem is divided into two regions: the no-investment and the investment region. When the pair $\{K(t), z(t)\}$ is inside the no-investment region, the agent consumes from the nondurable good, receives service flow from the current durable stock, but makes no new investment into the durable. In this case, the left part of the max is equal to zero. Also the agent chooses not to invest into the durable; hence the value of doing so must be negative. In other words, $J_z < J_K$.

On the other hand, when $\{K(t), z(t)\}$ is outside the no-investment region, the agent transfers nondurable into the durable good. Recall that the reverse transfer is assumed to be technologically impossible. At the time of investment, it must be that the value of investing into the durable is exactly equal to the value of the nondurable given up to do so. Hence we have that $J_z = J_K$ at times when investment takes place. It is also possible that $J_z > J_K$. That is the value of investing into the durable exceeds the value of nondurable given up to do so. In this case, agents would invest nondurable into the durable until J_z was equal to J_K.

The homogeneity of the problem allows us to write the value function as

$$J(K(t), z(t), t) = e^{-\rho t}\frac{z(t)^{A+B}}{A+B} g\left(\log\left(\frac{K(t)}{z(t)}\right)\right). \qquad (8)$$

This transformation reduces the problem to a time homogeneous problem in one state variable

$$\omega(t) \equiv \log\left(\frac{K(t)}{z(t)}\right). \qquad (9)$$

Expressing the state variables as a logarithm of the ratio of the capital stocks, rather than as the ratio directly, is done for analytical convenience.

The smooth-pasting condition $J_z = J_K$ holds at the boundary of the investment region. Given our single state variable the no-investment region will be given by $(-\infty, \omega^*]$, where ω^* needs to be determined as part of an agent's control problem. Agents invest nondurable into the durable good only when the ratio of nondurable to the durable good becomes sufficiently high.

At the investment boundary ω^* the smooth-pasting condition is

$$(e^{-\omega^*} + 1)g'(\omega^*) - (A+B)g(\omega^*) = 0 \tag{10}$$

The optimal nondurable consumption policy is given by

$$c_t = z_t \left(\frac{g'(\omega)}{Ae^{\omega}}\right)^{\frac{1}{A-1}} \tag{11}$$

This obtains from the usual envelope condition of dynamic programming, namely $U_c(c_t, z_t) = J_K(K_t, z_t, t)$. Notice that the optimal nondurable consumption to durable holdings ratio is a function of the state variable ω.

In the no-investment region, the left part of equation (7) evaluated at the optimal consumption c_t is

$$-(\theta(A+B)+\rho)g + (\theta+\mu)g' - (A-1)\left(\frac{g'}{Ae^{\omega}}\right)^{\frac{A}{A-1}} - \tfrac{1}{2}\sigma^2(g' - g'') = 0 \tag{12}$$

So far we have a second-order ordinary differential equation free boundary problem, and only one boundary condition. The free boundary occurs because we do not know ex-ante where ω^* should be.

Another boundary condition for the problem is the super-contact condition which must hold at the boundary of the no-investment region

$$J_{Kz} = J_{KK} \quad \text{or} \quad J_{zz} = J_{zK}.$$

These two equations turn out to be identical in the present problem. Together the smooth-pasting and the super-contact conditions help to uniquely determine the boundary ω^* of the no-investment region. The condition is a consequence of the fact that ω^* is chosen in a utility maximizing way. See Dumas (1991) for a discussion of these two conditions. Given the definition of J in (8), the super-contact condition is

$$((A+B)e^{\omega^*} + 1)g'(\omega^*) - (e^{\omega^*} + 1)g''(\omega^*) = 0. \tag{13}$$

Note that the condition only holds at ω^*.

The final boundary condition obtains when ω becomes very small. In the case where $A > 0, B > 0$, the value function must be equal to zero when $K = 0$. For $A < 0, B < 0$, the value function is equal to negative infinity when nondurable capital is zero. A more formal justification is provided in Theorem 4. These observations imply the following restrictions on $g(\cdot)$:

$$\lim_{\omega \to -\infty} g(\omega) = \begin{cases} 0 & \text{when } A > 0, B > 0, \\ \infty & \text{when } A < 0, B < 0. \end{cases} \tag{14}$$

Subject to (10), (13) and (14), the non-linear ordinary differential equation in (12) and the boundary of the no-investment region ω^* can be solved. Unfortunately a closed form solution to this problem does not exist. Instead the general problem can be solved numerically. Furthermore, it is possible to compute a closed-form approximate solution for $g(\cdot)$ by using perturbation analysis (see Appendix Section 8.7).

The following theorem (stated in a more rigorous way in the Appendix) confirms that as long as a sufficiently smooth solution to (12) exists, it will be equal to the value function of the problem. Furthermore, the theorem provides sufficient conditions for the optimality of the consumption process c_t and the investment process Φ_t.[6]

Theorem 1 *Assume that $\hat{g}$ and ω^* solve (12) subject to (10), (13) and (14). As long as $\hat{g}$ satisfies certain regularity conditions then*

$$J(K, z, t) \equiv e^{-\rho t} \frac{z^{A+B}}{A+B} \, \hat{g}\Big(\log(K/z)\Big)$$

gives the value function. Furthermore, assuming that the consumption process c_t^* (from 11) and an appropriately defined (see Appendix) investment process Φ_t^* exist, these will be the optimal controls.

3.2 The Optimal Investment Policy

The no investment region in this economy is characterized by a single number, ω^*. The log of the nondurables to durables ratio, or simply the nondurables to durables ratio, $\omega(t)$ is maintained by agents to be below ω^*. At any time τ when $\omega_\tau = \omega^*$, we will have $d\Phi > 0$ and the nondurable to durable ratio will be pushed back to the no-investment region. The size of the push will be infinitesimal, just big enough to prevent $\omega(t)$ from going above ω^*. Hence for all t it will be the case that $\omega(t) \in (-\infty, \omega^*]$. Furthermore, for any time t such that $\omega(t) \in (-\infty, \omega^*)$, new nondurable transfers into the durable, that is $d\Phi$, will be equal to 0. The three boundary conditions discussed in the previous section allow ω^* to be determined.

Using a generalized version of the Ito formula (for example, see Harrison (1990) or Karatzas and Shreve (1991)), we can show that for $\omega(t) = \log(K_t/z_t)$ we have

$$d\omega(t) = \mu_\omega(\omega(t))dt + \sigma dB - \frac{1 + e^{-\omega(t)}}{z_t} d\Phi_t, \tag{15}$$

$$\mu_\omega(\omega(t)) = \theta + \mu - \tfrac{1}{2}\sigma^2 - \frac{c_t}{K_t}. \tag{16}$$

Note from (11) that since the ratio of optimal nondurable consumption to the durable stock is a function of $\omega(t)$, the dynamics of $\omega(t)$ in the no-investment region depend only on $\omega(t)$, and not on

[6] A control problem similar to the one in this paper is analyzed in Shreve and Soner (1994). In their setup, they establish existence of the optimal consumption/investment policy.

K_t or z_t separately. However at the boundary, the singular component does depend on z_t. We can write the singular part of $\omega(t)$ as

$$\frac{1+e^{-\omega(t)}}{z_t}d\Phi_t = \left(\frac{1}{z_t}+\frac{1}{K_t}\right)d\Phi_t.$$

For a small transfer (using a Taylor series expansion) we have that

$$\begin{aligned}\log\left(\frac{K-d\Phi}{z+d\Phi}\right) &= \log(K-d\Phi)-\log(z+d\Phi)\\ &\approx \log(K)-\frac{1}{K}d\Phi-\log(z)-\frac{1}{z}d\Phi\\ &= \log\left(\frac{K}{z}\right)-\left(\frac{1}{z}+\frac{1}{K}\right)d\Phi \qquad (17)\end{aligned}$$

This is the origin of the singular term in the evolution of $\omega(t)$. The size of the nondurable transfer is just large enough so as to maintain $\omega(t)$ inside the no-investment region.

3.3 The Price of Durable Goods

The equilibrium price of a marginal unit of durable good can be computed in this economy as the agents' shadow price for that unit. Any of the identical agents may sell a unit of the durable to another agent. However since they are all at their optimal allocation, it must be that the (shadow) price clears the market by insuring that no trades take place. Recall that the technological assumption in the paper allows for a single unit of nondurable to be transferred into a single unit of durable good. This resulted in the $(-\infty,\omega^*]$ policy. We already see, therefore, that when $\omega(t)=\omega^*$ the price (in terms of units of the nondurable good) of the marginal unit of durable must be equal to 1. This would induce the same investment behavior at the boundary. Let us call $S(\omega(t))$ the price of the marginal unit of the durable good.

The following heuristic argument shows the intuition behind the shadow price definition. Theorem 2 provides a rigorous justification. For an infinitesimal purchase, we see that the shadow price must be given by

$$J(K-S\times dz, z+dz, t) = J(K,z,t) \qquad (18)$$

where the value function from (8) is given by $J(K,z,t)=e^{-\rho t}z^{A+B}/(A+B)g(\omega)$ where $\omega(t)=\log(K_t/z_t)$. Using a Taylor expansion, for a small investment we can write

$$J(K-S\times dz, z+dz,t) = J(K,z,t) - J_K(K,z,t)\times Sdz + J_z(K,Z,t)dz + O(dz^2). \qquad (19)$$

For a marginal investment into the durable good, satisfying the value matching condition in (18) requires that the dz terms in (19) equal zero. Solving for the $S(\omega(t))$ which guarantees this condition produces

$$S(\omega(t)) = \frac{J_z(K,z,t)}{J_K(K,z,t)}. \qquad (20)$$

Hence the price of the marginal durable unit is simply equal to the ratio of the marginal product of a unit of durable to the marginal product of a unit of nondurable.[7]

We can rewrite the durables price as

$$S(\omega(t)) = e^{\omega(t)}\left((A+B)\frac{g(\omega(t))}{g'(\omega(t))} - 1\right) \tag{21}$$

The smooth pasting condition (10) implies that

$$S(\omega^*) = 1$$

The price of a unit of durable at the investment boundary has to be simply 1 unit of nondurable in order to induce agents to make the same investment decisions as under the case of irreversibility.

Despite the fact that the nondurables to durables ratio contains a singular component, it turns out that the durable good price $S(\omega(t))$ does not.

Lemma 1 *The evolution of $S(t)$ can be written as*

$$dS_t = \mu_S(\omega(t))dt + \sigma_S(\omega(t))dB_t,$$

where $\sigma_S(\omega^*) = 0$. Notice that dS_t does not have a $d\Phi$ part.

3.4 The Short Rate and Bond Prices

The derivation of the short rate and bond prices in the economy uses essentially the same argument as in Cox, Ingersoll, and Ross (1985a). Imagine that agents solve the following problem

$$\begin{aligned}
\max_{c_t,x_t,y_t} \quad & \mathrm{E}_0\left[\int_0^\infty e^{-\rho t}u(c_t,z_t)dt\right], \\
\text{Such that} \quad dK_t &= x_tK(\mu dt + \sigma dB_t) + y_tK/P_t(dP_t + dD(t)) \\
&\quad +(1-x_t-y_t)Kr(t)dt - c_tdt - S_td\Phi_t + S_td\Xi_t, \\
dz_t &= -\theta z_tdt + d\Phi_t - d\Xi_t.
\end{aligned}$$

Here $P(t)$ is the ex-dividend price of some financial security, and $D(t)$ is the cumulative dividend paid by that security. S_t is the durable price in terms of the nondurable, $\Phi_t > 0$ is the continuous, cumulative investment process, and $\Xi_t > 0$ is the continuous, cumulative disinvestment process. Agents choose to invest a fraction x_t of their nondurable stock into the risky production technology, a fraction y_t of their nondurable stock into the financial asset, and a fraction $1 - x_t - y_t$ of their nondurable into a locally riskless investment with return $r(t)$. Agents may choose to consume out of their own nondurable holdings at a rate of c_t per unit time.

Keeping in mind the durable good price process, define the equilibrium of the economy as follows.

[7] In a similar setting, Kogan (2000) has shown that $S(\omega(t))$ is also the marginal value of Tobin's Q. In Kogan (2000) the value of a profit maximizing firm which makes irreversible investments into its production capacity is given by $S \times z$ where z is the amount of durable stock owned by that firm.

Definition 1 *Given an aggregate durable good process z_t^*, a constant returns to scale production technology with constant coefficients μ and σ, and a financial security with a cumulative dividend process of $D(t)$, the equilibrium of the economy is a collection of processes $\{r(t), P_t, S_t\}$ such that*

$$x_t = 1, \tag{22}$$

$$y_t = 0, \tag{23}$$

$$z_t = z_t^*. \tag{24}$$

Hence equilibrium requires that the total amount of lending and borrowing be zero, that the risky asset be in zero net supply, and that agents choose to make the same investments into the durable good as they would have made in the case of irreversible investment.

Given this definition of equilibrium, the following theorem is proved in the Appendix.

Theorem 2 *The equilibrium short rate and durable price are given by*

$$r(t) = \mu + \sigma^2 \frac{K J_{KK}}{J_K} \tag{25}$$

$$S_t = e^{\omega(t)} \left((A+B) \frac{g(\omega(t))}{g'(\omega(t))} - 1 \right) \tag{26}$$

Assuming that the cumulative dividend process can be written as $D(t) = \int_0^t a(t)P(t)dt$, the price of the financial security satisfies

$$-P_s(s) + (r(\omega) - a(\omega))P + (\sigma^2\gamma(\omega) - \mu_\omega(\omega))P_\omega - \frac{\sigma^2}{2} P_{\omega\omega} = 0 \tag{27}$$

subject to the appropriate boundary conditions. Here $\gamma(\omega_s) = -(J_{KK}K)/J_K = 1 - g''(\omega(t))/g'(\omega(t))$ is the time s derived risk aversion, and $\mu_\omega(\omega(t))$ is given by (16).

The price of a zero coupon bond which expires at time T is the solution of (27) with $a = 0$, subject the following boundary conditions

$$Z(\omega(T), T, T) = 1 \tag{28}$$

$$Z_\omega(\omega^*, \tau, T) = 0 \tag{29}$$

$$Z_\omega(-\infty, \tau, T) = 0 \tag{30}$$

The latter two conditions are necessary to rule out arbitrage opportunities (they follow from the requirement that the return on the asset at a reflecting boundary should be equal to the risk-free rate).[8] In fact, in the present model, condition (29) also guarantees that bond prices do not contain a singular component (i.e. dZ does not contain a $d\Phi$ term). It is convenient to define the price of a zero coupon bond in terms of the time left until it expires. Because of the time homogeneity of

[8] More specifically, for $Z = Z(\omega, t)$ applying Ito's lemma yields $dZ = Z_\omega d\omega(t) + Z_t dt + \frac{1}{2} Z_{\omega\omega} d\langle\omega\rangle$. Since $d\omega$ contains an infinite variation part, at a reflecting boundary we must have $Z_\omega = 0$. This implies that at the boundary we must have $Z_t + \frac{1}{2}\sigma^2 Z_{\omega\omega} = rZ$. This follows from the pricing equation (27) when $a = 0$ and when $P_\omega = 0$.

the economy, for any times s and τ

$$Z(\omega_s, \tau) = Z(\omega_s, s, s+\tau) \tag{31}$$

holds. Here τ is the time left until maturity. Given (31), we can define the yield of this zero as

$$y(\omega_s, \tau) \equiv -\frac{\log(Z(\omega_S, \tau))}{\tau} \tag{32}$$

Hence $y(\omega_s, \cdot)$ gives us the term structure of interest rates in this economy. Notice that the entire term structure curve depends on ω_s as long as the prices of zero coupon bonds depend on ω_s.

4 Base Cases of the Model

This section discusses some limiting cases of the present model. First, the cases of consumption over only durables and only nondurables are treated. Second, closed form lower and upper bounds for the value function of the general problem are provided.

4.1 Separating Durable and Nondurable Consumption

In order to build some intuition about the paper's results consider the two limiting cases (durable-only or nondurable-only consumption) of the present model. In the case of only nondurable consumption ($B = 0$), the present model reduces to the Cox, Ingersoll, and Ross (1985a) equilibrium model, where there is only a single constant returns to scale production technology. The latter assumption of constant coefficients implies that the short rate in the model is constant. Therefore the term-structure of interest rates is constant as well.

In the current model, when $B = 0$ agents do not care about service flow from the durable good. In this case, the value function and optimal consumption policy are given by

$$J(K_t, t) = e^{-\rho t} C_0^{1-A} \frac{K_t^A}{A}, \tag{33}$$

$$c(K_t) = C_0^{-1} K_t, \tag{34}$$

$$C_0 = \frac{A-1}{A\mu - \rho + \frac{1}{2}(A-1)A\sigma^2}. \tag{35}$$

This is a well know result (see, for example, Merton (1990)). The implied risk aversion from the value function is simply equal to the coefficient of relative risk aversion from the utility function, and both are given by $\gamma = 1 - A$. The interest rate, from (25), is given by

$$r(t) = r = \mu - (1-A)\sigma^2. \tag{36}$$

In this case, the interest rate is constant. As a comparative static, it decreases with risk-aversion and with the variability of the production process. As people are either more risk averse or as there is more risk in the economy, the alternative to investing in the risky production process must become less desirable in order for the bond market to clear. Similarly as the return on the production technology increases, the short rate also must increase to prevent people from borrowing to invest into production. Implied yields from bond prices are always equal to r.

The case of preferences for only durable service flow has been studied by Hindy and Huang (1993). Their model is equivalent to the one in this paper when $A = 0$. While this is a substantially more complicated problem than the $B = 0$ case, they were able to obtain a closed form solution.[9] The following theorem states the relevant results

Theorem 3 *In the present model, with $A = 0$, the value function is given by*

$$\begin{aligned} J(K_t, z_t, t) &= e^{-\rho t}\frac{z_t^B}{B}\left(b_0\Omega^k + b_1\right) && (37)\\ b_0 &= \frac{b_1 B(\Omega^*)^{1-k}}{k(1+\Omega^*) - B\Omega^*} && (38)\\ b_1 &= \frac{1}{\theta B + \rho} && (39)\\ k &= \frac{-(\theta+\mu-\frac{1}{2}\sigma^2) + \sqrt{2(\theta B+\rho)\sigma^2 + (\theta+\mu-\frac{1}{2}\sigma^2)^2}}{\sigma^2} && (40)\\ \Omega^* &= -\frac{1-k}{B-k} && (41) \end{aligned}$$

where $\omega(t) \equiv K_t/z_t$. The no-investment region is given by $[0, \Omega^*]$.

The proof is available from the author upon request, or see Hindy and Huang (1993). Notice that the implied risk aversion in this case is again constant and is given by

$$\gamma \equiv -\frac{J_{KK}K}{J_K} = 1 - k \tag{42}$$

This is not equivalent to the relative risk aversion of the utility function itself (this would be $1 - B$). The quantity

$$\theta + \mu - \tfrac{1}{2}\sigma^2 \tag{43}$$

is the drift rate of $\omega(t) = \log(K_t/z_t)$.

The short rate is given by (25) and is constant

$$r(t) = r = \mu - (1-k)\sigma^2 \tag{44}$$

The introduction of consumption over service flow does substantially change the results of the nondurable consumption case. In particular, the implied risk aversion is no longer equal to the risk aversion of the utility function, and instead depends on the parameters of the model. However, the implied risk aversion remains a constant. Hence the short rate is constant, and all implied yields will equal the short rate.

[9] Interestingly, Jonathan Ingersoll has independently solved this same problem. However, he has never published the derivation.

Using an argument similar to the one used to derive (21) and using the form of the value function given in (37), we see that the durable price in this case is given by

$$S_t = \frac{\Omega(k - 1 + \Omega^{*k}\Omega^{-k})}{\Omega^* k} \tag{45}$$

Here $\Omega^* = (k-1)/(B-k)$, and we see that when $\omega(t) = \Omega^*$ the durable price is equal to 1. Also in order to guarantee $\Omega^* > 0$, the parameters of the problem must be chosen such that $B < k < 1$ (this assumes that $B < 1$ is an exogenously imposed constraint). Given this relationship it is easy to check that $dS/d\Omega > 0$. Hence the durable price is increasing in $\omega(t)$ and it approaches its upper limit of 1 as the economy approaches the investment point Ω^*.

4.2 Bounds for the Value Function

Though it is difficult to solve for the value function in closed form in the general case of the model, it is possible to solve for closed form bounds on the value function. An upper bound can be obtained by considering the case where durable investment is perfectly reversible. The details of the derivation are not presented here (they are available from the author, and a similar derivation can be found in Damgaard, Fuglbjerg, Munk 2000). The value function turns out to be of the following form

$$J(K+z,t) = Ce^{-\rho t}\frac{(K+z)^{A+B}}{A+B},$$

for some constant C.[10] No distinction is made between the durable and nondurable stocks because of the perfect reversibility assumption. The nondurable consumption rate and the durable holdings are given by constants times the current wealth (i.e. $K+z$). Hence z responds instantaneously to changes in K. The implied risk aversion of investors is equal to $1 - A - B$ and is a constant. Hence interest rates are also constant. A drawback of this upper bound is that as $K \to 0$, the value function simply becomes $J(z,t)$, and we learn little about the asymptotic behavior of the value function of the constrained problem.

Another upper bound, as well as a lower bound, can also be determined in closed form. These bounds turn out to be extremely useful as they characterize the behavior of the actual value function as nondurable capital goes to zero. The following theorem states the relevant results.

Theorem 4 *A lower bound J_{low} for the value function of problem (6) is given by*

$$J_{low}(K,z,t) = e^{-\rho t}\frac{z^{A+B}}{A+B}C_{low}^{1-A}e^{\omega A}, \tag{46}$$

where $\omega = \log(K/z)$ and $C_{low} = (A-1)/(A\mu - \rho - B\theta + \frac{1}{2}(A-1)A\sigma^2)$. An upper bound is given

[10] More specifically, if we write the optimal durable holdings as $\zeta(K+z)$, and let

$$f(C,\zeta) \equiv 2\frac{\zeta^B}{A+B}\left(\frac{(A+B)C}{\zeta^B A}\right)^{\frac{A}{A-1}} - 2C\left(\left(\frac{(A+B)C}{\zeta^B A}\right)^{\frac{1}{A-1}} - \mu + \zeta(\theta+\mu)\right) - 2\frac{C\rho}{A+B} - (1-A-B)C(1-\zeta)^2\sigma^2.$$

Then C and ζ solve the following two nonlinear algebraic equations: $f(C,\zeta) = 0$ and $f_\zeta(C,\zeta) = 0$.

by

$$J_{high}(K,z,t) = e^{-\rho t}\frac{z^{A+B}}{A+B}C_{high}^{1-A}(1+e^{\omega})^{B}e^{\omega A}, \tag{47}$$

where $C_{high} = (A-1)/(A\mu + B\mu - \rho)$. *Furthermore, we have*

$$\lim_{\omega \to -\infty} J(K,z,t) = O(e^{\omega A}).$$

Comparing (46,47) to equation (8), and using the limiting result of the theorem, provides a rigorous justification for the boundary conditions for $g(\cdot)$ in (14).

It is instructive to go through the part of the proof dealing with the lower bound (the derivation of the upper bound is given in the Appendix). Consider the optimal nondurable consumption policy of an investor with preferences given by (6), but who is constrained to never invest into or disinvest from the durable good (i.e. $d\Phi = 0$). We can rewrite this investor's instantaneous utility function as follows

$$e^{-\rho t}u(c,z(t)) = e^{-\rho t}\frac{c^{A}z(t)^{B}}{A+B} = \frac{A}{A+B}z(0)^{B}e^{-(\rho+\theta B)t}\frac{c^{A}}{A},$$

where we have used the fact that with no investment $z(t) = \exp(-\theta t)z(0)$. Upto a constant this is exactly the same utility function and problem as in the $B = 0$ case discussed previously, with the time discount factor replaced by $\tilde{\rho} \equiv \rho + \theta B$. Using the results of (33) and (35), the form of J_{low} follows immediately. J_{low} is a lower bound because the strategy of never investing into the durable is available to the unconstrained agent as well. Hence the optimal policy must be no worse than the $d\Phi = 0$ one.

We see, therefore, that in the case of nonseparable preferences over durables and nondurables, but with no ability to change the durable stock, the investor's problem becomes exactly the same as in the case of no durable consumption at all, with the time discount factor modified to take into account the rate of depreciation of the existing durable stock. This observation gives us an intuition for a stronger limiting result than the one given in the Theorem: as the capital stock goes to 0, whether or not the agent is able to invest into the durable does not matter (hence only nondurable consumption matters) which renders the lower bound exactly equal to the value function of the general problem. While this is true for the numerical solution of the model and in the case of the perturbation analysis when B is small (see Lemma 2), without having a tighter upper bound than the one in (47) it is difficult to prove this conjecture in general. Hence we settle for the weaker result given in the Theorem.

As will be shown in the next section, the introduction of the realistic assumption that economic agents care about both durable and nondurable consumption (i.e. $A \neq 0$ and $B \neq 0$) causes the implied risk aversion to change in response to changes in capital stocks, which in turn causes interest rate fluctuations. It should be stressed that in the present framework in order for risk aversion and interest rates to be functions of the state of the economy, consumers must have preferences over both durables and nondurables, and durable investment must be irreversible. Without either of these two assumptions interest rates and risk aversion would be constant.

5 Analysis of the General Case

This section discusses the durable goods investment process, and the joint behavior of interest rates and consumption in the general case of the model. Unfortunately, the general case of the economy in this paper can not be solved explicitly. However, an approximate closed form solution is available by expanding around the base economy with $B = 0$ (i.e. only nondurable consumption matters). The case of B being small seems to be justified from the empirical papers of Dunn and Singleton (1986) and Ogaki and Reinhart (1998). Section 8.1 of the Appendix contains a discussion of reasonable choices for the parameter values.

5.1 Durable Investment

Recall that the no-investment region in the economy is given by $(-\infty, \omega^*]$. The state variable, $\omega = \log(K/z)$ measures how much capital stock exists relative to the durable stock in the economy. When this ratio is sufficiently high, or $\omega = \omega^*$, new investment takes place. Since investment into the durable good is irreversible, agents do not transfer the nondurable stock into the durable stock until the nondurable stock is sufficiently plentiful. The appropriate measure of "plentifulness" in this context is the ratio of the two durable stocks.

At the point of new investment, $d\Phi$ units of nondurable capital are instantaneously transferred into the durable stock. The size of the transfer is just large enough to maintain ω in the no-investment region $(-\infty, \omega^*]$. The cumulative transfer from the nondurable to the durable stock is

$$\Phi(t) = \int_0^t d\Phi(t).$$

Figure 2 shows the behavior of ω and Φ for a single realization of uncertainty in the economy.

As can be seen from the Figure, $\Phi(t)$ (the increasing line in the graph) increases only at those times when ω hits the upper bound ω^*. Note that in between investment times, the durable good stock depreciates at a rate of θ.

5.2 Nondurable Consumption

Durable goods affect interest rates in the model through the effect which durable investment has on the consumption of nondurables. From (11), we see that the consumption rate is given by

$$c_t = z_t \left(\frac{g'(\omega)}{Ae^{\omega}} \right)^{\frac{1}{A-1}}.$$

Noticing that $e^{\omega(t)} = K_t/z_t$ we can express consumption as a fraction of the nondurable

$$\frac{c(t)}{K(t)} = \left(\frac{g'(\omega)e^{-A\omega}}{A} \right)^{\frac{1}{A-1}}. \tag{48}$$

Therefore the nondurable consumption to nondurable capital ratio is a function only of $\omega(t)$.

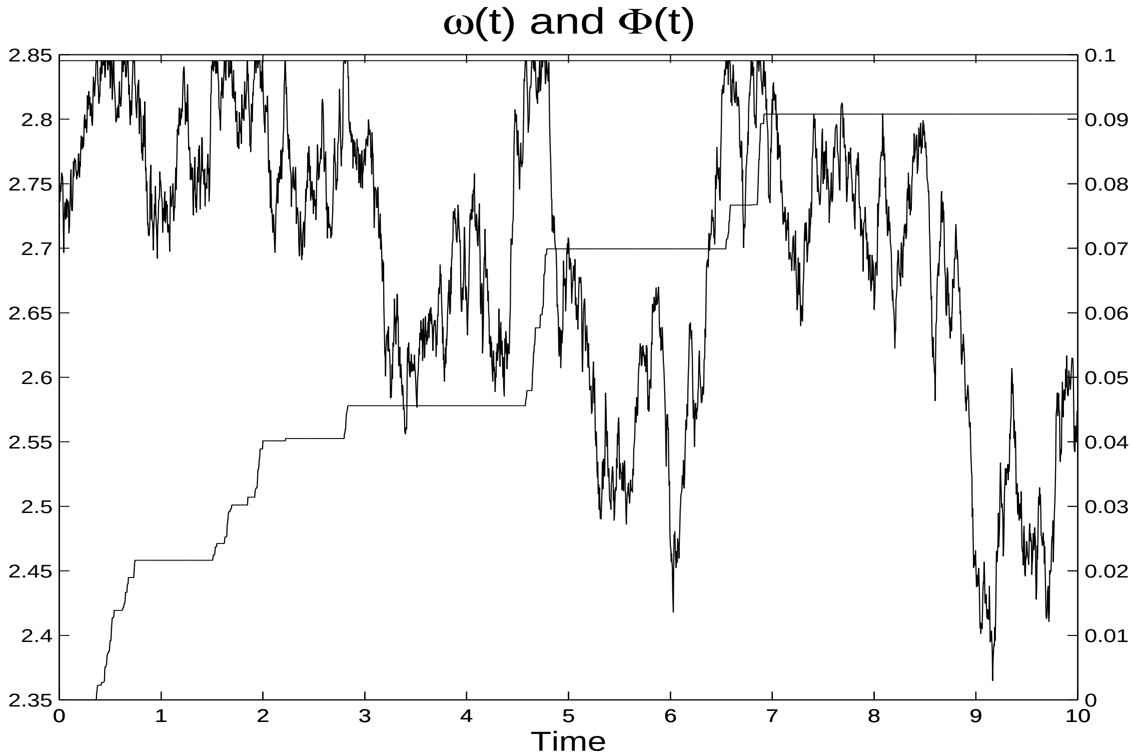

Figure 2: The figure shows $\omega(t)$ and $\Phi(t)$ for a single realization of uncertainty in the economy.

For small values of B the consumption to nondurable ratio is given by the following lemma.

Lemma 2 *The consumption to nondurable ratio is given by the following lemma.*

$$\frac{c_t}{K_t} = C_0^{-1} + \frac{\theta}{1-A}B - \frac{e^{A(k_1-1)(\omega(t)-\omega^*)}k_1}{(1-A)AC_0(1+A(k_1-1))(k_1-1)}B + O(B^2), \tag{49}$$

where ω^* is the boundary of the no-investment region and is a constant, and where

$$k_1 = \frac{-\theta + \frac{\mu-\rho}{A-1} + \frac{1}{2}(1+A)\sigma^2 + \sqrt{2(A\theta+\rho)\sigma^2 + (-\theta + \frac{\mu-\rho}{A-1} + \frac{1}{2}(1+A)\sigma^2)^2}}{A\sigma^2}. \tag{50}$$

Under the following parameter restrictions

$$\begin{aligned} \rho &> \rho_{min} \equiv A\mu + \tfrac{1}{2}(A-1)A\sigma^2, & (51)\\ \theta + \mu - \tfrac{1}{2}\sigma^2 &> 0, & (52)\\ A &> -\frac{\theta+\mu-\frac{1}{2}\sigma^2}{\sigma^2}. & (53) \end{aligned}$$

for $A > 0$ we have that $k_1 \geq 1$, and for $A < 0$ we have $k_1 \leq 1$. In particular, $A(k_1-1) \geq 0$. Also if $A < 0$ there exists a non-empty region $[\rho_{min}, -A\theta]$ such that if $\rho \in [\rho_{min}, -A\theta]$, we will have $0 \leq k_1 \leq 1$, and for $\rho > -A\theta$, we will have $k_1 < 0$.

Notice that the term

$$\theta + \frac{\mu - \rho}{1 - A} - \tfrac{1}{2}(1 + A)\sigma^2$$

is the drift rate of $\omega(t) = \log(K_t/z_t)$ in the case where $B = 0$. We therefore see the connection between k_1 and the k which obtained in Theorem 3.

Before interpreting the results of the Lemma, a note is in order on conditions (51),(52), and (53). Condition (51) is necessary to insure that $C_0 > 0$, or that consumption as a fraction of nondurable capital is positive in the base case of the model when $B = 0$. Condition (52) is a mild condition which insures that the economy tends on average to the durable investment point ($\theta + \mu - \frac{1}{2}\sigma^2$ is the drift of $\log(K/z)$ when durable and nondurable consumption are both zero). Finally, condition (53) says that the representative agent should not be overly risk averse, and is necessary for the approximate solution to be valid.

From equation (49) we see that as $B \to 0$, the consumption to capital ratio approaches its constant value in the base case of no durable consumption (see equation (34)). The second term in (49) adjusts for the fact that durables and nondurables may be substitutes or complements in the present framework. It is straightforward to check that

$$\operatorname{sgn}\left(\frac{\partial^2 u(c,z)}{\partial c\,\partial z}\right) \;=\; \operatorname{sgn}(A) \;=\; \operatorname{sgn}(B).$$

Hence if $B > 0$, the two goods are complements, and if $B < 0$ the two goods are substitutes. In the case where they are complements, the ratio of consumption to nondurables increases relative to the base case. Since the durable stock depreciates over time (hence the presence of the θ term), the marginal utility over nondurable consumption will be lower in the future, and agents will consume a larger fraction of the nondurable stock today. In the case of substitutability, an incentive exists to defer nondurable consumption to the future when the durable stock will be lower, and when the marginal utility of nondurable consumption will be higher.

The dependence of the third term in (49) on $\omega(t)$ indicates that the consumption over nondurables ratio depends on the state of the economy. Note that in the base case of no durable consumption, the ratio c/K is constant. From the Lemma, we see that $A\,(k_1 - 1)$ is always positive. This allows us to make some statements about the behavior of the third term. When the economy is far away from the investment point (i.e. when $\omega(t)$ is small), the third term goes to zero, and c/K becomes approximately constant. In fact, when $\omega = -\infty$, c/K is exactly equal to the optimal consumption in the case of the lower bound of Theorem 4 (i.e. when no durable investment is allowed). For small B, this confirms the conjecture made in the discussion following Theorem 4.

Considering the signs of the following terms

$$\begin{aligned} C_0 &> 0, \\ 1 - A &> 0, \\ A(k_1 - 1) &> 0, \end{aligned}$$

the sign of the third term in (49) is given by the sign of $k_1 \times B$. If $B > 0$, then $k_1 > 1$ and proximity to the investment point decreases c/K. More generally, that is for $B < 0$, the sign of the effect depends on whether k_1 is positive or negative. If k_1 is negative, then c/K is decreasing as ω^* approaches, and if k_1 is positive, then c/K increases. This effect is at the heart of the results in the paper. As is shown in the Lemma, the sign of k_1 is negative if $A\theta + \rho > 0$. Recalling from (5) that $A = \delta(1-\gamma)$, this condition can be rewritten as

$$\frac{1}{\gamma} > \frac{\theta\delta}{\theta\delta + \rho}. \tag{54}$$

$1/\gamma$ is the intertemporal elasticity of substitution (IES), or a measure of the willingness of consumers to postpone consumption over time. Equation (54) states that if the IES is sufficiently high, then agents will decrease c/K as they approach the durables investment point. If the IES is low, then c/K will increase when durable investment is imminent. This IES condition may hold or not hold for reasonable values of the model parameters (see Appendix section 8.1 for a discussion).

The intution for this result is rather straightforward. When investment into the durable good becomes imminent, agents who are sufficiently willing to substitute consumption over time (i.e. with a high IES) will forego present nondurable consumption in order to invest into the durable good sooner (which is accomplished by lowering c/K, thereby increasing the growth rate of $\omega(t)$) – people close to buying a house may choose to defer going out to dinner in order to save money for the house. However, if an agent's IES is low, then lowering present nondurable consumption to accelerate the purchase of the durable good is not optimal. Instead, the higher relative amount of the nondurable capital causes agents with low IES's to consume at a higher rate out of the nondurable stock.

Note that the above effect has to do with the willingness to postpone consumption, rather than with agents' aversion to uncertainty. This is most easily seen by noting that in the limit as $\sigma \to 0$ (i.e. as the production technology becomes riskless), the sign of k_1 is still determined by condition (54). Also, though the discussion has so far focused on the approximate closed form solution, the same results hold in the numerical solution of the model. The one difference is that the threshold point at which c/K goes from increasing to decreasing is close to, but not exactly equal to, the value in (54).

5.3 Risk Aversion

This significance of the discussion in the previous section lies in the effect that changes in c/K have on the implied risk aversion of agents in the economy. Interest rates in the model are set so as to clear the lending market in the nondurable good, and as such depend on the willingness of agents to bear risk in the production technology. Hence the appropriate measure of implied risk aversion in the economy is given by

$$\gamma(\omega(t)) \equiv -\frac{KJ_{KK}}{J_K}.$$

This is a measure of the sensitivity of an agent's value function to fluctuations in K. In the base case of the model with only nondurable consumption (i.e. when $u = c^{1-\gamma}/(1-\gamma)$), we have the well known result that the implied risk aversion is a constant given by $\gamma(\omega(t)) = \gamma$.

Locally, changes in the capital stock effect the agent only through their effect on nondurable consumption. Hence changes in c/K ought to impact the sensitivity of agents to production shocks in the nondurable stock. When c/K is decreasing in $\omega(t)$, it is also decreasing in K (since $om(t) = \log(K(t)/z(t))$). Hence a 1% positive shock to the productive nondurable capital will increase nondurable consumption by less that 1% (since c/K will fall). Likewise, a 1% fall in the nondurable stock will result in a smaller than 1% fall in nondurable consumption (since c/K will rise). Hence when c/K decreases as the investment point of the economy approaches, nondurable consumption is less sensitive to shocks in the productive capital, and agents become less risk averse (that is $\gamma(\omega(t))$ is decreasing in $\omega(t)$).

An analogous argument shows that $\gamma(\omega(t))$ should be increasing if c/K increases as the durable investment point approaches. The reason is that a 1% shock to the nondurable capital stock will result in more than a 1% changes in nondurable consumption.

To make these statements concrete define the capital stock elasticity of nondurable consumption as follows

$$\eta(\omega(t)) \equiv \frac{\partial c/c}{\partial K/K}. \tag{55}$$

From the form of nondurable consumption in (11) and from the form of the value function in (8), it is easy to show that the implied risk aversion of agents in the economy is (upto a constant) equal to the elasticity of nondurable consumptiona with respect to the nondurable capital stock, or

$$\gamma(\omega(t)) = 1 - g''(\omega(t))/g'(\omega(t)) = (1-A) \times \eta(\omega(t)). \tag{56}$$

When $\eta \approx 1$, which happens when $\omega(t)$ is low, the implied risk aversion is equal to its value in the only nondurable economy. As $\omega(t)$ increases, implied risk aversion either falls (if the IES is high) or rises (if the IES is low). Whether $\gamma(\omega(t))$ rises or falls with $\omega(t)$ provides one of the major testable implications of the model. This is discussed in greater length in the empirical part of the paper.

For small values of B, the following lemma confirms the conclusions of the above discussion. Recall that $A(k_1 - 1) \geq 0$.

Lemma 3 *The implied relative risk aversion is given by*

$$\gamma(\omega(t)) = (1-A) - \frac{k_1 e^{A(k_1-1)(\omega(t)-\omega^*)}}{1 + A(k_1-1)} B + O(B^2) \tag{57}$$

where k_1 is given in (50), and ω^* is the constant boundary of the no-investment region.

The proof is given in the Appendix. Condition (54) determines whether the implied risk aversion is decreasing or increasing with $\omega(t)$.

5.4 The Term Structure

Equation (25) gives the instantaneous risk-free interest rate in the model

$$r(t) = \mu - \gamma(\omega(t))\sigma^2.$$

Note that μ and σ, the drift and volatility of the production technology, are constant by assumption. Hence the short rate is highest when the implied risk aversion $\gamma(\omega(t))$ is lowest, and is lowest when $\gamma(\omega(t))$ is highest. The only difference between this result and the analogous result in the base case of the model (either with $B = 0$ or $A = 0$) is that the implied risk aversion is a function of the state of the economy.

From the preceeding discussion, we see that the size of the IES (from equation (54) is the determining factor of whether the short rate is highest or lowest at the point when new durable investment takes place. When the IES is high, the short rate is highest at the point of new investment. When the IES is low, the short rate is lowest at the point of new durable investment. Figure 3 shows an example of both cases.

The term structure reflects expectations about future short rates. Hence the shape of the term structure depends on the behavior of the short rate around the point of new durables investment. When the short rate is high (for high IES), the term spread should be downward sloping at the point of new investment, as future short rates have to be lower than the present one (from equation (25)). When the short rate is lowest at the point of new durable investment (for low IES), the term spread should be upward sloping. Figure 4 shows an example of both cases.

In the present model, the drift and volatility of the production technology are assumed to be constant. Clearly this ignores one important source of variation in actual interest rates. However, to the extent that changes to the production opportunity set are persistent, they ought to affect short and long rates by the same amount. Hence focusing on the slope of the term spread, rather than on the absolute level of individual rates, helps to cancel out the effects of changes in the production set, and thus provides a cleaner test of the implications of durable investment and changing risk aversion for interest rates.

Another interesting affect on the term structure comes from the rate of depreciation θ of the durable good. When θ is low, the term structure is relatively flat. This is because with a low depreciation rate, the growth rate of $\omega(t) = \log(K/z)$ is small. Hence the current short rate is relatively persistent. However, for a large value of θ, the durable stock depreciates quickly, causing the growth rate of $\omega(t)$ to be large. Hence the term structure is steeply sloped, reflecting the relatively rapid anticipated movement of the economy towards the investment point. Figure 4 shows the term structure for a high value of θ.

5.5 Discussion of Results

It is obvious from the figures presented so far that this model does a poor job of matching some salient moments of the actual data. For example, the variation in interest rates generated by the model is far too small to be empirically plausible. Also the level of rates is too high. Consider U.S.

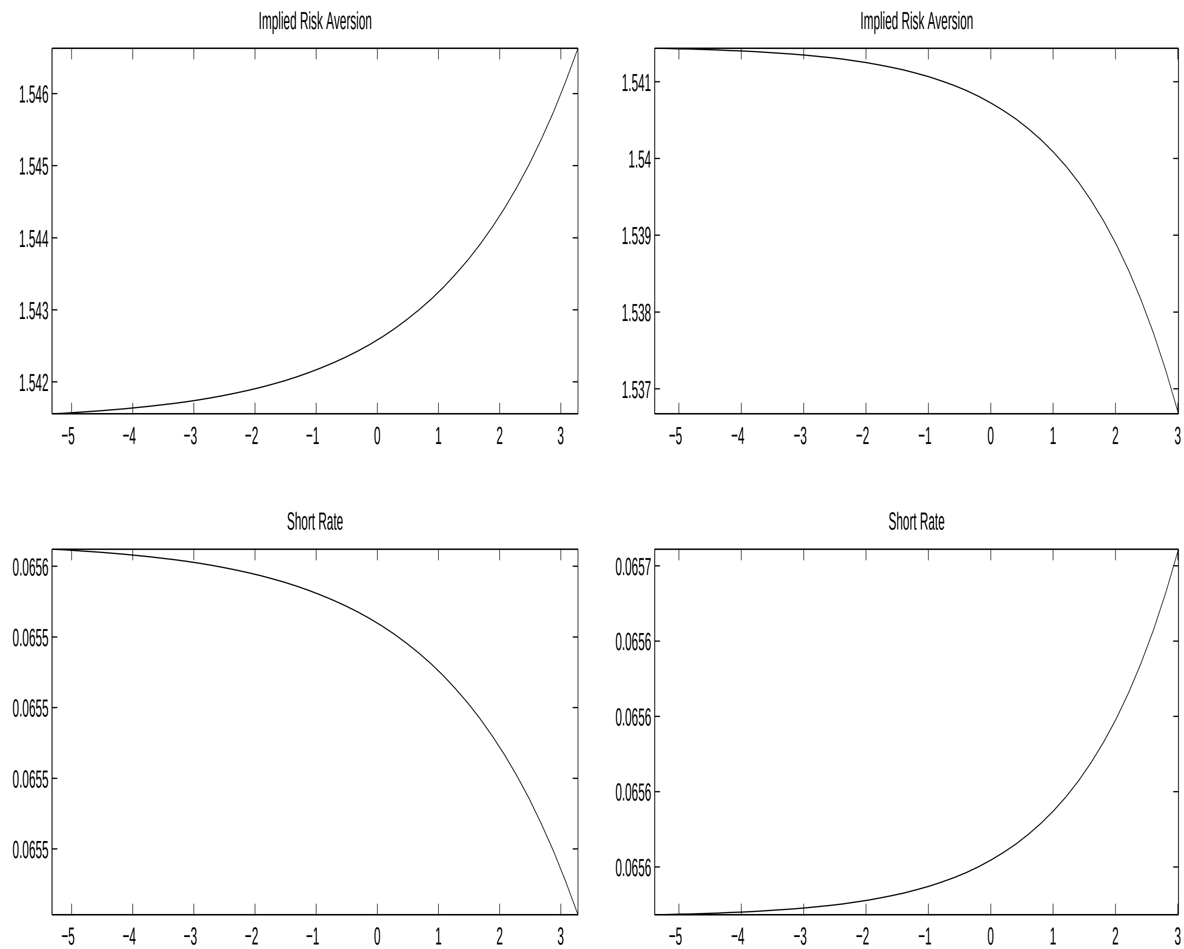

Figure 3: The figures show $r(\omega(t))$ and the implied risk aversion coefficient $\gamma(\omega(t))$. The parameter values are $\mu = 0.0970$, $\sigma = 0.1428$, $\theta = 0.1$, $A = -0.54$, $B = -0.06$. The left figures have $\rho = 0.05$, and the right figures have $\rho = 0.07$.

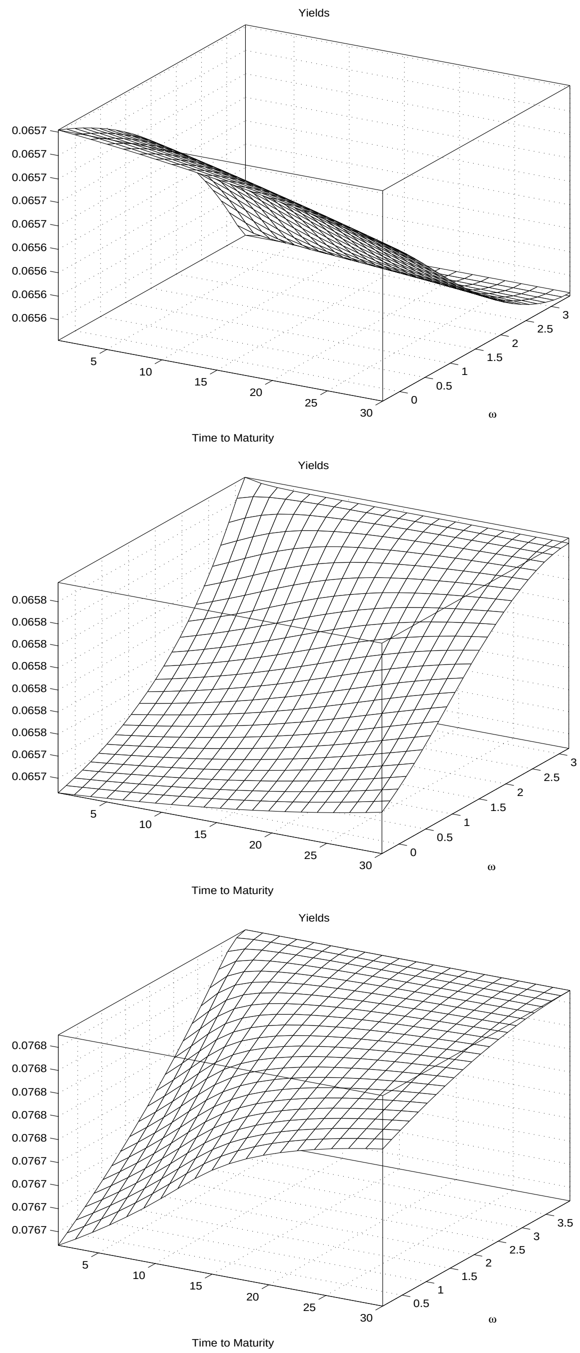

Figure 4: The figures shows the term structure as a function of $\omega(t)$. The parameters for the top two figures are given by $\mu = 0.0970$, $\sigma = 0.1428$, $\theta = 0.1$, $A = -0.54$, $B = -0.06$. The top figure $\rho = 0.05$, the middle figure has $\rho = 0.07$. The bottom figure has $\mu = 0.0970$, $\sigma = 0.1428$, $\theta = 0.3$, $A = -0.01$, $B = -0.0011$, $\rho = 0.07$.

nominal interest rate data. From 1947–1999, the monthly term spread in U.S. data has averaged 1.04% with a standard deviation of 1.15%. Over this time period, the annual real short rate (30 day yield) in the U.S. has averaged 1.09% with a standard deviation of 2.68%.

These features of the model fall under the general category of the equity premium puzzle. Mehra and Prescott (1985) pointed out that models with time separable, CRRA utilities are unable to match the equity premium and consumption process observed in the U.S. in the 20th century. This observation has effected a fairly large literature which has taken aim at this puzzle. Prominent papers include Constantinides (1990), Abel(1990), Epstein and Zin (1989,1991), and Campbell and Cochrane (2000). Campbell (1999,2000) and Sundaresan (2000) contain summaries of the literature.

Most relevent to the issue at hand is the work of Epstein and Zin (1989,1991) which shows that the degree of risk aversion can be separated from the intertemporal elasticity of substitution. In the present model, when the IES is increased (by increasing $A + B$) the spread in short rates becomes much higher. This is intuitive since the change in the nondurable consumption to capital ratio is higher when individuals have higher IES. However, increasing the IES in the present model also means that the instantaneous risk aversion of agents decreases, and hence the level of interest rates rises (from its already overly ambitious level). Hence a separation of IES from risk aversion would maintain a more reasonable level of interest rates, while at the same time achieving a larger spread between short rates in different states of the world.

Notice, on the other hand, that the spread of the term structure in the present model can be made quite reasonable given the variation in short rates. Then the depreciation rate of the durable is fairly high, the term structure in the present model is rather steep (as can be seen Figure 4). Of course, the low and high points of the term structure must lie between the extreme short rates possible in the model. Since these are quite low, the level of term spread in the model is also low. However, the relative size of the term spread (relative to $\max(r) - \min(r)$) can be quite substantial.

Additionally two important sources of variation in actual nominal interest rates have been ignored, namely inflation and changes in the production opportunity set. Since durable investment is related to expectations of future investment opportunities in the production technology (e.g. durable investment is more likely to happen when J_K, and hence production prospects, are low) we may expect that a model which took this into account would produce effects similar to those discussed in this paper, but of a larger magnitude – that is, variation in short rates would be higher.

These observations suggest that the present model should not be thought of as providing an appropriate econometric specification for interest rate and consumption data. Rather the model points out a potentially important effect which durable investment has on risk aversion, and therefore on interest rates. Despite the model's inability to match certain moments of the data, the effect which it suggests is likely to be quite robust. Hence the model may point out an important economic interaction, while being unable to succesfully account for all relevant moments of the actual data.

Table 1: Summary of correlation signs.

	IES High	IES Low
Corr(D, ND)	$-$	$+$
Corr(D, S)	$-$	$+$
Corr(ND, S)	$+$	$+$

6 Empirical Results

The focus of the empirical tests is on the implications of condition (54). Assuming that this condition holds, we will have that nondurable consumption as a fraction of nondurable capital should decrease as durable investment becomes imminent. The restrictions which this places on consumption and interest rates are

- Durable investment and the nondurable consumption to nondurable capital ratio ought to be negatively correlated.
- Durable investment should be negatively related to the term spread.
- The ratio of nondurable consumption to nondurable capital should be positively related to the term spread.

On the other hand, if the IES condition in (54) does not hold, the restrictions implied by the model are

- Durable investment and the nondurable consumption to nondurable capital ratio ought to be positively correlated.
- Durable investment should be positively related to the term spread.
- The ratio of nondurable consumption to nondurable capital should be positively related to the term spread.

Note that while the two durable correlations change signs in the two cases, the correlation of nondurable consumption with the term spread is always positive. Table 1 summarizes these implications.

As has already been mentioned, the reason that the term spread, rather than the level of the short rate, is the quantity of interest is because interest rate levels are affected by changes in the production opportunity set. However, to the degree that such changes are persistent, this effect will get cancelled out when looking at differences in rates.

Before tests of these implications are carried out, several remaining empirical issues need to be addressed. The first of these is how the consumption to capital stock ratios can be measured. The problem with these ratios is that consumption is measured as a flow (e.g. the consumption in a given quarter), whereas the durable stock is measured at a particulat point in time. In turns out that it is easier to use a flow measure which is itself related to the capital stock. In the model, the instantaneous output (or GNP) of the economy is given by[11]

$$dY = K(\mu dt + \sigma dB).$$

The ratio of instantaneous nondurable consumption to output is therefore

$$ND/GNP \equiv \frac{c\,dt}{dY} = \frac{c/K\,dt}{\mu dt + \sigma dB}.$$

The ratio of instantaneous durable investment to output is

$$D/GNP \equiv \frac{d\Phi}{dY} = \frac{\text{Constant} \times dl}{\mu dt + \sigma dB}.$$

This follows from equation (82) in the Appendix.[12] Note also that dl is nonzero only at those times when invsetment into the durable good takes place. Hence the ratio of nondurable consumption to GNP provides a (noisy) measure of c/K, and the ratio of durable investment to GNP provides a measure of the proximity of the economy to the investment point, and furthermore this last measure does not depend on the current level of the stock of productive capital. Since data on nondurable and durable consumption, and on GNP, are of the flow variety, these ratios are observable (albeit not instantaneously, but at a quarterly frequency).

A critical aspect of the empirical analysis is knowledge of the direction of the IES condition in (54). This, of course, is not directly observable. However, the correlation of ND/GNP and D/GNP provides a noisy signal of the direction of the IES condition. If the IES is sufficiently large, then $\text{Corr}(D, ND)$ should be negative; otherwise the correlation should be positive. In fact, a test of the implications of Table 1 is to check whether $\text{sgn}(\text{Corr}(D, ND)) = \text{sgn}(\text{Corr}(D, S))$, and whether $\text{sgn}(\text{Corr}(ND, S)) = 1$. Figure 1 is supportive of the first implications, as are more formal tests (presented below).

A caveat is in order about the interpretation of the statistical tests. The fact that $\text{Corr}(D, ND)$ may switch signs implies that certain parameter values, which the model assumed to be constant, are in fact changing over time. As long as the time variation in these parameters is sufficiently slow, then the results of the model may not be materially impacted over the medium term. However during actual transition times from one sign regime to another, the model has little to say about the data because the time variation in the parameters may be extremely important even over the short term. This suggests that a better test of the implications of the model would use cross sectional international data. Since different countries may be in different parameter regimes, the implications of the model could be tested without assuming that parameter values are changing over time. Unfortunately, international data are hard to obtain, hence the tests in this model use time series data from the U.S.

[11]This follows from the usual accounting identity that output is equal to investment plus consumption, which in the model translates to $dY = dK + cdt + d\Phi = K(\mu dt + \sigma dB)$.

[12]The process $l(t)$ is the maximum distance through time t by which the ratio $\log(K/z)$ would have exceeded the investment point ω^* if no durable investment had been allowed. Details are given in the Appendix.

6.1 Data Description

Quarterly data on U.S. durables, nondurables, and output are readily available. I use the standard NIPA (National Income and Product Accounts) data set available from the U.S. Department of Commerce. Durable consumption should include all investment which will (1) result in future service flow for investors, and (2) will not result in future revenue streams (as could be obtained from renting out a room in one's house). I therefore define durable investment as the sum of personal consumption expenditures on durable goods and gross private domestic investment into residential real estate. Nondurable consumption consists of consumer expenditures on nondurables and services. All time series are converted into real terms using the NIPA chain-type quantity indexes, with 1996 as the base line year. Seskin and Parker (1998) provide details of how these indexes are constructed and used.

The ratio $D/GNP(t)$ is real durable investment in quarter t as defined above over that quarter's real GNP. The series $ND/GNP(t)$ is similarly defined.

Monthly data on government interest rates are obtained from Ibbotson Associates. Long term bond returns and yields are from the U.S. Intermediate Term government bond series, and the short term returns and yields are from the U.S. 30 day treasury series. The term spread $SPREAD(t)$ in a given month is the difference between the long and the short yield.

The data used in the analysis are at a quarterly frequency. Such data are available in the NIPA data set starting from 1947. The interest rate data in a given quarter is the average over the quarter's 3 months. Furthermore, all analysis is performed with detrended values of the three variables D/GNP, ND/GNP, and $SPREAD$. The detrending is intended to reduce spurious correlations which may be due to long term trends in these variables.

Correlations between the three variables D/GNP, ND/GNP, and $SPREAD$ are computed in rolling windows. The results in the paper are insensitive to window sizes of 12–20 quarters. The time series of $\text{Corr}(ND, S)$ and $\text{Corr}(D, S)$ is computed within each window between contemporaneous values of the variables. On the other hand, $\text{Corr}(ND, D)$ is computed within each window between D/GNP and a value of ND/GNP which is lagged by one quarter. The reason for this timing convention is that actual durables investment is associated with certain fixed costs (not modelled) which induces even more lumpiness in durables purchases than is allowed for in the model. Hence durables purchases may discontinuously decrease the nondurable stock and hence affect the contemporaneous ratio of c/K (which cannot happen in the model since investment is a continuous process). However, since ND/GNP is persistent in reality and in the model, a measure of $\text{Corr}(ND, D)$ which is insensitive to fixed costs can be construcuted by using lagged values of ND.

Before proceeding with the analysis, a note on the distinction between nominal and real interest rates is in order. The majority of empirical literature (Harvey (1988) is an exception) on interest rates and the business cycle examines the relationship between nominal interest rates (e.g. the nominal term spread) and real economic activity (e.g. real GNP). The general finding is that there is a relationship between real output and both contemporaneous and lagged measures of nominal interest rates. Since nominal rates contain real rates as well as expectations about future inflation, the observed relationship may be attributable to either source. There has been some work (e.g. Plosser and Rouwenhorst (1994)) which suggests that the empirical findings are attributable to the information which nominal rates contain about real interest rates, and not about future inflation. To the extent that the correlations which exist between consumption measures and nominal interest rates are attributable to correlations between consumption and real rates, the effects in this paper are applicable.

6.2 Data Analysis

Figure 5 shows a time series of the quarterly values of D/GNP, ND/GNP, and $SPREAD$ from 1947–1999. Keep in mind that the two consumption ratios are in real terms, and the term spread is in nominal terms. The durables to GNP ratio is seen to be strongly procyclical, indicating that durables investment as a share of GNP is high in economic peaks, which is what may have been expected from the analysis in this paper: since durables investment is irreversible, it occurs after periods of high economic growth. The nondurables share of GNP, on the other hand, is countercyclical. This is a result of consumption smoothing by investors, and is consistent with a countercyclical (on average) ratio of c/K. Finally, the term spread is seen to be countercyclical on average, which is consistent with past empirical work (e.g. an almost identical graph appears in Fama and French (1989)).

Hence in the postwar U.S. economy, on average, the durables to GNP is procyclical, and nondurables to GNP and the term spread are countercyclical. The two latter implications are consistent with the the model as long as the IES condition in (54) holds (i.e. IES is sufficiently high). However, the test of the mechanism proposed in this paper is to see whether the signs of the correlations between these three variables follow the two regimes outlined in Table 1.

Figure 1 provides a preliminary answer to this question. As can be seen, the signs of $\mathrm{Corr}(D,S)$ and of $\mathrm{Corr}(ND,D)$ do have a similar time series pattern. Hence to the degree that $\mathrm{Corr}(ND,D)$ proxies for the direction of the IES condition in (54), the signs of durable investment and the term spread do seem to behave in accordance with Table 1.

To perform more rigorous tests of the implications of Table 1, consider the following regression

$$SPREAD(t) = \beta_0 + \beta_1 ND/GNP(t) + \epsilon(t).$$

As can be seen from Table 2, β_1 is positive and significant. This is consistent with the implication of the model that the nondurable consumption to capital ratio should be unconditionally positively correlated with the term spread. On the other hand, consider the following regression

$$SPREAD(t) = \beta_0 + \beta_1 D/GNP(t) + \epsilon(t).$$

The explanatory power of the regression is far lower, and the coefficient is not significant. This is again consistent with the implication of the model that the correlation of durables investment with the term spread depends on the sign of the IES condition in (54).

To allow for the possibility that the sign of $\mathrm{Corr}(D,S)$ may indeed be a time varying function

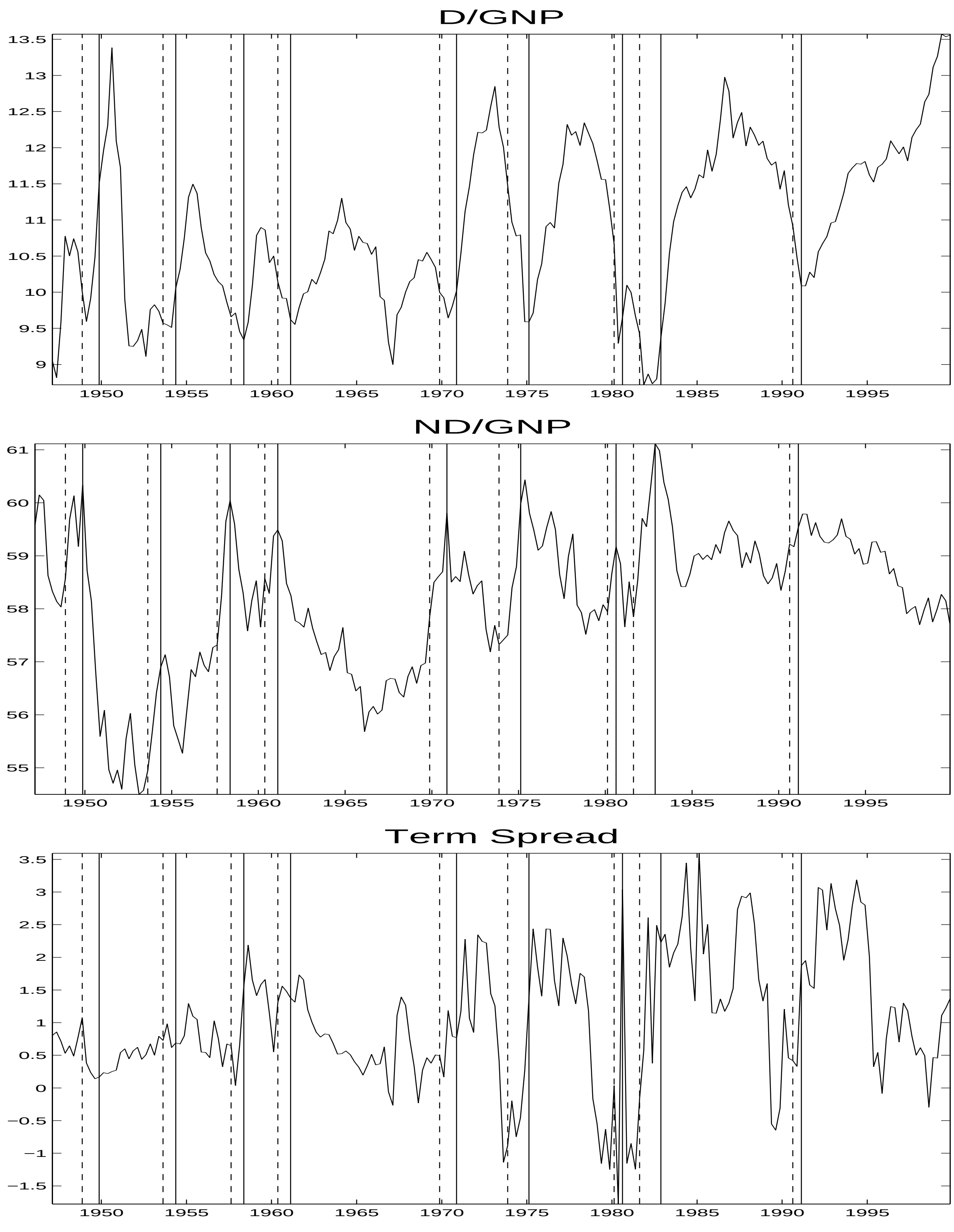

Figure 5: The figures show the durable consumption to GNP ratio, the nondurable consumption to GNP ratio, and the term spread. The consumption and GNP data are in real terms. The dashed vertical lines are NBER business cycle peaks, and the solid vertical lines are the troughs.

of the model parameters, consider the following regression

$$SPREAD(t) = \beta_0 + \beta_1 D/GNP(t) + \beta_2 D/GNP(t) \cdot \mathbf{1}[\text{Corr}(ND, D) > 0] + \epsilon(t).$$

Hence the sign of the effect which durables investment has on the term spread is allowed to depend on whether the IES condition holds (β_1) or whether it does not hold ($\beta_1 + \beta_2$). As can be seen from Table 2, the ability to condition on the current economic regime, vastly improves the explanatory power of the regression. Furthermore, both β_1 and β_2 become significant. The fact that β_1 is significantly negative suggests that durable investment is associated with a negative term spread when the IES is sufficiently high. And the fact that $\beta_1 + \beta_2$ is significantly positive suggests that the opposite correlation applies when the IES is sufficiently low. This directly confirms the model's implications.

As a robustness check, we perform the same conditioning for the ND/GNP regression, that is

$$SPREAD(t) = \beta_0 + \beta_1 ND/GNP(t) + \beta_2 ND/GNP(t) \cdot \mathbf{1}[\text{Corr}(ND, D) > 0] + \epsilon(t).$$

As can be seen from Table 2, the explanatory power of the regression and the estimate of β_1 do not change, whereas β_2 is insignificantly different from zero. This result supports the idea that the correlation of the term spread with nondurable consumption has the same sign regardless of the IES regime.

7 Conclusion

This paper has argued that consumption of durable and nondurable goods is an important determinant of interest rates in the economy. By solving a general equilibrium production economy, the paper was able to identify an important economic interaction between durable investment, risk aversion, and interest rates. The main channel for this effect is the irreversibility of durable investment, which causes agents to adjust their nondurable consumption patterns as investment into the durable good seems imminent. Depending on the intertemporal elasticity of substitution, this change in nondurable consumption may increase or decrease the implied risk aversion of investors by changing their sensitivity to shocks in the production technology.

An empirical investigation using U.S. interest rate and consumption data seems to support the implication of the model that the correlation of durable investment with the term spread depends on the magnitude of the intertemporal elasticity of substitution in the economy.

The theoretical and empirical results of this paper suggest that durable consumption is an important ingredient of any asset pricing model which will ultimately succeed in describing the joint behavior of asset prices and consumption in the economy. However, the present model suffers from the "equity premium puzzle," in that it is unable to produce a level and a degree of variation in the short rate which are consistent with the data. Two promising avenues of research which may overcome this problem are (1) a model of nondurable and durable consumption using recursive preferences so as to separate the effects of risk aversion and intertemporal substitution, and (2) a model which allows a time varying investment opportunity set. As is usually the case the issue of

Table 2: Regressions use quarterly data from 1951–1999. All regressions are run using values of the regressors lagged by one quarter as instruments. The Corr(ND, D) variable is computed in 16 quarter windows, and corresponds to the 16 quarter window ending in and including quarter t. T-statistics are computed using Newey-West standard errors, and are reported in parentheses.

$$SPREAD(t) = \beta_0 + \beta_1 ND/GNP(t) + \epsilon(t)$$

β_1		R^2
0.5842		0.3249
(6.4402)		

$$SPREAD(t) = \beta_0 + \beta_1 ND/GNP(t) + \beta_2 ND/GNP(t) \cdot \mathbf{1}[\text{Corr}(ND, D) > 0] + \epsilon(t)$$

β_1	β_2	R^2
0.5583	0.1094	0.3270
(5.4002)	(0.3296)	

$$SPREAD(t) = \beta_0 + \beta_1 D/GNP(t) + \epsilon(t)$$

β_1		R^2
-0.1526		0.0175
(-1.3743)		

$$SPREAD(t) = \beta_0 + \beta_1 D/GNP(t) + \beta_2 D/GNP(t) \cdot \mathbf{1}[\text{Corr}(ND, D) > 0] + \epsilon(t)$$

β_1	β_2	R^2
-0.5440	1.0290	0.2016
(-4.1408)	(6.3957)	

how investor heterogeneity affects the results is an interesting, though difficult, question.

Finally a validation of the empirical results of this model using international data would provide further evidence in support of the robustness of the economic mechanism proposed in this paper.

8 Appendix

8.1 Choice of Parameter Values

Dunn and Singleton (1986) estimate a model with a representative consumer with the same preference assumptions as those made in this paper. While in their model the assumption about service flows from durable and nondurable consumption are different than those made in this paper, their parameter estimates are nevertheless a good starting point.

I use the results given in Table 2 of the Dunn and Singleton paper. Note that the parameter γ in this paper is $1 - \gamma_{DS}$, where γ_{DS} is the gamma parameter in the Dunn and Singleton paper. A representative 95 percent confidence interval for γ is 2 ± 2, and for δ the confidence interval is 0.9 ± 0.2. Dunn and Singleton's point estimate for the time preference parameter ρ is negative, though positive values are within the 95 percent confidence interval. They find that $\exp(-\rho/12) = 1.003 \pm 0.006$. The high end of this confidence interval is $\rho = 0.036$. Finally, their interval for the monthly depreciation coefficient for the durable stock is 0.995 ± 0.04. Roughly this translates to a range for θ given by $[0, 0.55]$ (negative values are not allowed). The point estimate $0.995 = \exp(-\theta/12)$ translates to $\theta = 0.06$. Other authors report a much higher value of θ. For example, Ogaki and Reinhart (1998) use a value of θ of around 0.2 (bottom of page 1084).

Using point estimates, the IES condition in (54) holds

$$\frac{1}{\gamma} > \frac{\theta\delta}{\theta\delta + \rho} \quad \Leftrightarrow \quad 0.5 > \frac{0.06 \times 0.9}{0.06 \times 0.9 + \rho}$$

as long as $\rho > 0.054$. This is outside the confidence interval for ρ found by Dunn and Singleton. However since their point estimate for ρ is negative the validity of their results with regard to ρ are not clear. Clearly there are values of the parameters inside the confidence intervals which will allow the IES condition to hold (for example, choose $\gamma = 1.6$, $\delta = 0.9$, $\theta = 0.1$ and $\rho = 0.07$).

I calibrate the production technology to match the mean and volatility of the SP500 index from 1947–1999. Over this time period, the real annual return on SP500 has averaged 0.10392 with a volatility of 0.16811 (data obtained from Ibbotson Associates). Monthly log returns on the production technology are $\exp((\mu - 0.5\sigma^2)/12 + \tilde{B})$, with $\tilde{B} \sim N(0, \sigma^2/12)$, implying estimates for μ and σ of 0.0970 and 0.1428 respectively.

8.2 Lemmas 4 and 5

Lemma 4 establishes sufficient conditions for a given function to dominate the value function of an agent. Lemma 5 establishes sufficient conditions for an optimal control, and for a given function to be equal to the value function. These lemmas will be used in the proof of Theorem 1.

Lemma 4 *Assume that there exists a function $\hat{J}(K, z, t)$ which is continuously differentiable in all its arguments and twice continuously differentiable with respect to the first argument, and which satisifies the following growth condition*

$$|J(K, z, t)| \leq D_1 \left[1 + \left(\sqrt{K^2 + z^2}\right)^{D_2}\right], \tag{58}$$

for some time dependent constants D_1 and D_2. Let us further assume that $\hat{J}$ satisfies the differential inequality

$$\max[\mathcal{D}\hat{J} + \hat{J}_t + e^{-\rho t}u(c, z), \hat{J}_z - \hat{J}_K] = 0 \tag{59}$$

where

$$\mathcal{D}[f(K, z, t)] = (\mu K - c)f_K + \tfrac{1}{2}\sigma^2 K^2 f_{KK} - \theta z f_z.$$

For a policy $\{c, \Phi\}$ to be admissible, we require that the following regularity conditions be satisfied

$$
\begin{aligned}
c(t) &\leq D_3(1 + \sqrt{K^2 + z^2}), && (60)\\
E\left[\Phi(t)^{2m}\right] &\leq D_4, && (61)
\end{aligned}
$$

for a constant D_3 and a time dependent constant D_4. Furthermore, $\{c, \Phi\}$ must satisfy the following integrability condition

$$E_t\left[\int_t^\infty e^{-\rho t}\left|u\Big(c(s), z(s)\Big)\right|ds\right] < \infty. \tag{62}$$

Also $\hat{J}$ satisfies the following boundary condition for all admissible policies

$$\lim_{T\to\infty} \mathrm{E}_t\left[\hat{J}(K(T), z(T), T)\right] = 0. \tag{63}$$

Furthermore, the following regularity condition holds for all T

$$\int_t^T \Big(\sigma K(s)\hat{J}_K(s)\Big)^2 ds < \infty \quad a.e. \tag{64}$$

Then we will have that

$$E_t\left[\int_t^\infty e^{-\rho t}u(c(s), z(s))ds\right] \leq J(K(t), z(t), t) \leq \hat{J}(K(t), z(t), t). \tag{65}$$

Proof. Let $\hat{J}(\cdot)$ be the candidate value function. Set some arbitrary time $T > t$. From the generalized Ito's lemma and the dynamics of $K(t)$ and $z(t)$ given in (2–3) we can write

$$
\begin{aligned}
\int_t^T e^{-\rho t}u(c(s), z(s))ds + \hat{J}(T) = \\
\int_t^T e^{-\rho t}u(c(s), z(s))ds + \hat{J}(t) + \int_t^T [\mathcal{D}\hat{J}(s) + \hat{J}_s(s)]ds \\
+ \int_t^T [\sigma K(s)\hat{J}_K(s)]dB_s + \int_t^T [\hat{J}_z(s) - \hat{J}_K(s)]d\Phi_s.
\end{aligned} \tag{66}
$$

where the differential operator $\mathcal{D}[\cdot]$ is given above. Since $\hat{J}$ satisfies (59) and given that $d\Phi(t) \geq 0$ we see that (66) implies

$$\int_t^T e^{-\rho t}u(c(s), z(s))ds + \hat{J}(T) \leq \hat{J}(t) + \int_t^T [\sigma K(s)\hat{J}_K(s)]dB_s. \tag{67}$$

Let us define the stopping time τ_n as

$$\tau_n \equiv \inf\{s \geq t : \int_t^T \Big(\sigma K(s)\hat{J}_K(s)\Big)^2 ds = n\}.$$

From condition (64), we know that $\tau_n \to T$ a.s. when $n \to \infty$. Also we know that

$$\int_t^{\tau_n} [\sigma K(s)\hat{J}_K(s)]dB_s$$

is a martingale (see Lemma V.5.1 in Fleming and Rishel (1975)). Taking expectations in (67) we therefore can write

$$\mathrm{E}_t\left[\int_t^{\tau_n} e^{-\rho t}u(c(s),z(s))ds + \hat{J}(\tau_n)\right] \leq \hat{J}(t).$$

Using condition (62), we can apply Lebesgue dominated convergence to show that

$$\mathrm{E}_t\left[\int_t^{\tau_n} e^{-\rho t}u(c(s),z(s))ds\right] \rightarrow \mathrm{E}_t\left[\int_t^{T} e^{-\rho t}u(c(s),z(s))ds\right].$$

Using the fact that $\hat{J}$ has polynomial growth (condition (58)), and the regularity conditions on c and Φ in (60,61), we can use an argument very similar to the one in Fleming and Rishel (1975) Theorem V.5.1 to conclude that

$$\mathrm{E}_t\left[\hat{J}(\tau_n)\right] \rightarrow \mathrm{E}_t\left[\hat{J}(T)\right].$$

Hence we conclude that

$$\mathrm{E}_t\left[\int_t^{T} e^{-\rho t}u(c(s),z(s))ds + \hat{J}(T)\right] \leq \hat{J}(t).$$

Taking the limit as $T \rightarrow \infty$, and using boundary condition (63) and the consumption condition in (62), we conclude from Lebesgue dominated convergence that

$$\mathrm{E}_t\left[\int_t^{\infty} e^{-\rho t}u(c(s),z(s))ds\right] \leq \hat{J}(t),$$

which confirms the original claim. Q.E.D.

Lemma 5 *Assume that a function $\hat{J}$ satisfies all of the conditions of Lemma 4. Assume that there are controls $c^*(t)$ and $\Phi^*(t)$ such that the following conditions hold:*

$$\begin{aligned} \mathcal{D}^*\hat{J}(t) + \hat{J}_t(t) + e^{-\rho t}u(c^*(t),z^*(t)) &= 0, & (68)\\ \int_t^T [\hat{J}_z(t) - \hat{J}_K(t)]d\Phi^*(t) &= 0, & (69) \end{aligned}$$

where $\mathcal{D}^*$ is the Ito operator under the consumption process c^*. Then we will have

$$\hat{J}(K(t),z(t),t) = J(K(t),z(t),t)$$

In other words $\hat{J}$ will be equal to the value function. Furthermore, $c^*(t)$ and $\Phi^*(t)$ will be the optimal controls.

Proof. The proof proceeds exactly as in the case for Lemma 4. The exception is that in light of conditions (68–69) we can replace the inequality in (67) with an equality. The remaining arguments are identical to Lemma 4. From (65) we see that $\hat{J} = J$. Q.E.D.

In order to prove Theorem 1 we will verify that the smooth-pasting and super-contact boundary conditions imply that the conditions in Lemmas 4 and 5 are satisfied. This will show that the control policy proposed in the paper is indeed optimal.

8.3 Proof of Theorem 1

The idea of this proof is to show that a function $\hat{J}$, given by the solution to an appropriately defined free boundary problem, satisfies the conditions of Lemma 4. Then Lemmas 4 and 5 can be invoked to verify that $\hat{J}$ is equal to the value functions, and that the associated controls are optimal.

Theorem 5 *Assume that a function $h(\omega)$ satisfies*

$$-(\theta(A+B)+\rho)h+(\theta+\mu)h'-(A-1)\left(\frac{h'}{Ae^{\omega}}\right)^{\frac{A}{A-1}}-\frac{1}{2}\sigma^2(h'-h'')=0 \tag{70}$$

subject to the boundary conditions that

$$\begin{aligned}
(e^{-\omega^*}+1)h'(\omega^*)-(A+B)h(\omega^*) &= 0 && (71)\\
((A+B)e^{\omega^*}+1)h'(\omega^*)-(e^{\omega^*}+1)h''(\omega^*) &= 0 && (72)\\
\lim_{\omega\to-\infty} h(\omega) &= \begin{cases} 0 & \textit{when } A>0, B>0,\\ \infty & \textit{when } A<0, B<0.\end{cases} && (73)
\end{aligned}$$

where ω^* is a free boundary which satisfies

$$\begin{aligned}
e^{\omega^*} > \; & \frac{1}{Bc_3}\Big[(-1+A+B)c_2+(-1+A)c_3 \\
& -\sqrt{2B((-2+2A+B)c_1+c_2-Ac_2)c_3+((-1+A+B)c_2+(-1+A)c_3)^2}\Big]
\end{aligned} \tag{74}$$

where

$$\begin{aligned}
c_1 &= \theta(A+B)+\rho\\
c_2 &= (A+B)(\theta-\mu)+2\rho\\
c_3 &= -2\rho+(A+B)(2\mu-(1-A-B)\sigma^2)
\end{aligned}$$

where the parameters are assumed to be such that $c_3<0$. Furthermore, we assume that

$$\operatorname{sgn}\left[(A+B)h(\omega)-(1+e^{-\omega})h'(\omega)\right]=-\operatorname{sgn}[A] \tag{75}$$

when $\omega<\omega^*$, and that there exist finite constants c_4, c_5 such that

$$h'(\omega)\,e^{-A\omega}\in[c_4,c_5]. \tag{76}$$

Let us define a function $g(\cdot)$ as follows

$$g(\omega)=\begin{cases} h(\omega) & \textit{if } \omega<\omega^*\\ \left(\frac{1+e^{\omega}}{1+e^{\omega^*}}\right)^{A+B} h(\omega^*) & \textit{if } \omega\ge\omega^*\end{cases} \tag{77}$$

Then the value function for the agent's problem in (6) is given by

$$J(K(t),z(t),t)\equiv e^{-\rho t}\frac{z(t)^{A+B}}{A+B}g(\omega(t)) \tag{78}$$

where

$$\omega(t)\equiv\log\frac{K(t)}{z(t)}$$

Furthermore the optimal control will be to maintain the ratio $\log(K/z)$ *below* ω^* *at all times, with the control* $\Phi(t) \in \mathbf{X}^+$ *being exerted only when* $\log(K/z) = \omega^*$.

Proof. The proof will proceed in three steps. First we want to show that when $\omega > \omega^*$ we will have that

$$\begin{aligned} \mathcal{D}J(t) + J_t(t) + e^{-\rho t}u(c^*(t), z^*(t)) &\leq 0 \\ J_z - J_K &= 0 \end{aligned}$$

Secondly we want to show than when $\omega \leq \omega^*$ we have that

$$\begin{aligned} \mathcal{D}J(t) + J_t(t) + e^{-\rho t}u(c^*(t), z^*(t)) &= 0 \\ J_z - J_K &\leq 0 \end{aligned}$$

Once this has been established we can invoke Lemma 4. Finally (thirdly) we will state the form of the control policy which will satisfy Lemma 5.

Part 1. Using the boundary conditions (71–72) and the ODE in (70) we can show that when $\omega = \omega^*$ the value of $h(\omega^*)$ is given by

$$h(\omega^*) = \left[\frac{1}{2(-1+A)(1+e^{\omega^*})^2} \left(\frac{A+B}{A+Ae^{\omega^*}} \right)^{\frac{A}{1-A}} (-2c_1 - 2e^{\omega^*}c_2 + e^{2\omega^*}c_3) \right]^{-1+A} \tag{79}$$

When $\omega > \omega^*$ we want to show that for all $z(t)$

$$\sup_{c(t)} \mathcal{D}J(t) + J_t(t) + e^{-\rho t}u(c^*(t), z(t)) \leq 0 \tag{80}$$

Using the definition of $g(\cdot)$ we see that when $\omega > \omega^*$ the left hand side of the above equation is given by $(A+B)^{-1}$ times

$$\frac{1}{(1+e^{\omega})^2}(-2c_1 - 2e^{\omega}c_2 + e^{2\omega}c_3) - 2(-1+A)\left(\frac{A+B}{A+Ae^{\omega}} \right)^{\frac{A}{-1+A}} \left(\frac{1+e^{\omega}}{1+e^{\omega^*}} \right)^{\frac{A+B}{-1+A}} h(\omega^*)^{\frac{1}{-1+A}}$$

Using the definition of $h(\omega^*)$ above we can rewrite this as

$$\frac{1}{(1+e^{\omega})^2}(-2c_1 - 2e^{\omega}c_2 + e^{2\omega}c_3) - \left(\frac{1+e^{\omega}}{1+e^{\omega^*}} \right)^{\frac{B}{-1+A}} \frac{1}{(1+e^{\omega^*})^2}(-2c_1 - 2e^{\omega^*}c_2 + e^{2\omega^*}c_3)$$

It is clear that this is equal to 0 when $\omega = \omega^*$. We would like to show that the derivative of the above expression with respect to ω is negative when $\omega > \omega^*$ and $A > 0$, and is positive when $\omega > \omega^*$ and $A < 0$. That would confirm (80). Dividing both sides by $\left(\frac{1+e^{\omega}}{1+e^{\omega^*}}\right)^{\frac{B}{-1+A}}$, differentiating with respect to ω, and collecting terms we find that we need for the following to hold

$$-2(-2+2A+B)c_1 - 2(1-A)c_2 + (-c_2(-2+2A+2B) + 2(1-A)c_3)e^{\omega} + Bc_3(e^{\omega})^2 \leq 0$$

Since $c_3 < 0$ by assumption, this is a concave function in e^{ω} if $B > 0$, and is a convex function in e^{ω} if $B < 0$. Solving for the roots, we see that the larger root is given by (74). Hence if e^{ω^*} satisfies the condition in (74), this expression will be negative for all $e^{\omega} > e^{\omega^*}$ when $B > 0$, and positive for all $e^{\omega} > e^{\omega^*}$ when $B < 0$.

Given the definition of $g(\cdot)$ it is easy to check that whenever K and z are such that $\omega > \omega^*$ we have that

$$J(K, z, t) = \left(\frac{z+K}{1+e^{\omega^*}}\right)^{A+B} \frac{h(\omega^*)}{A+B},$$

which is the value function if the agent chooses to invest enough nondurable into the durable to immediately drive K/z down to e^{ω^*}. From this it is clear that $J_z = J_K$.

Part 2. When $\omega > \omega^*$ it is clear that from the definition of $h(\cdot)$ the Bellman type condition that

$$\mathcal{D}J(t) + J_t(t) + e^{-\rho t}u(c^*(t), z^*(t)) = 0$$

is satisfied. Also condition (75) ensures that $J_z < J_K$ whenever $\omega < \omega^*$. This follows from the definition of $J(K, z, t)$.

Note first that condition (76) implies that for $a > b$ we have

$$h(a) - h(b) < \frac{e^{-Aa} - e^{-Ab}}{A} \times [c_4, c_5]. \tag{81}$$

From the form of $J(\cdot)$ in (78), the dynamics of K and z, and conditions (76,81) it is straightforward to check that the polynomial condition (58), the regularity conditions (60,61), the consumption condition in (62), the transversality condition in (63) and the integrability condition in (64) all hold for the candidate policy. Therefore, using Parts 1 and 2 of the Theorem, we see that the function J as defined in (78) dominates the value function of the problem (by Lemma 4), and the candidate controls are indeed optimal (by Lemma 5).

Part 3. In order to satisfy Lemma 5 we see that is it sufficient to maintain $\log(K/z)$ below ω^* at all times, and to make sure that $c(t)$ is chosen in compliance with (70). That is the condition $U_c(c, z) = J_K(K, z, t)$ must hold. Furthermore, to satisfy (69) the control $d\Phi$ can be non-zero only when $\omega = \omega^*$, at which point it should be just large enough so as to maintain continuity of $\Phi(t)$. Let us define $\hat{K}(t)$ and $\hat{z}(t)$ as the unregulated nondurable and durable good processes which would result if $d\Phi(t) = 0$ for all t. Let us define a regulation process $l(t)$ as follows

$$l(t) \equiv \sup_{s \leq t} \left[\log\left(\frac{\hat{K}(s)}{\hat{z}(s)}\right) - \omega^*\right]^+ .$$

It is easy to verify that $l(t) \in \mathbf{X}^+$. We will define the regulated process $\omega(t)$ as follows

$$\log\left(\frac{K(t)}{z(t)}\right) = \log\left(\frac{\hat{K}(t)}{\hat{z}(t)}\right) - l(t).$$

Keeping in mind the dynamics of K and z given in (2–3), and applying Ito's lemma to both sides of this equation we have

$$d\left[\log\left(\frac{\hat{K}(t)}{\hat{z}(t)}\right) - l(t)\right] = \left(\theta + \mu - \tfrac{1}{2}\sigma^2 - \frac{c(t)}{K(t)}\right) dt + \sigma dB(t) - dl(t),$$

(note that c/K is a function of ω which only needs to be defined inside the no-investment region $[-\infty, \omega^*))$ and

$$d\log\left(\frac{K(t)}{z(t)}\right) = \left(\theta + \mu - \tfrac{1}{2}\sigma^2 - \frac{c(t)}{K(t)}\right)dt + \sigma dB(t) - \left(\frac{1}{z(t)} + \frac{1}{K(t)}\right)d\Phi(t).$$

From this we conclude that $\frac{z(t)+K(t)}{K(t)z(t)}d\Phi(t) = dl(t)$. Since $dl(t) \neq 0$ only when $K(t)/z(t) = e^{\omega^*}$ we get that the optimal durable investment policy is

$$\Phi^*(t) = \frac{1}{1+e^{\omega^*}}\int_0^t K(s)dl(s) \tag{82}$$

Q.E.D.

8.4 Proof of Theorem 2

The interest rate and the differential equation in (27) follows from the standard Cox, Ingersoll, and Ross arguments. Given the price of a unit of durable in (26) we need to show that the amount of durable good held by the trader in an economy with irreversible investment is indeed optimal. The nondurable and durable processes can be written as

$$\begin{aligned} dK_t &= \mu K dt + \sigma K dB_t - c_t dt - S_t d\Phi_t + S_t d\Xi_t & (83)\\ dz_t &= -\theta z_t dt + d\Phi_t - d\Xi_t & (84)\end{aligned}$$

where Φ_t and Ξ_t are elements of $\mathbf{X}^+$. In other words, transferring nondurable into durable is accomplished via the process $d\Phi_t$ and transferring durable into nondurable is accomplished via the process $d\Xi_t$. Both of these are positive by virtue of the fact that Φ_t and Ξ_t are elements of $\mathbf{X}^+$. A unit of durable can be transferred into S_t units of nondurable, and vice versa whenever an agent chooses to do so.

We will repeat the arguments in Lemma 4. Applying the generalized Ito's lemma to the value function, we have

$$\begin{aligned} &\int_t^T e^{-\rho t}u(c(s),z(s))ds + \hat{J}(T) = \\ &\int_t^T e^{-\rho t}u(c(s),z(s))ds + \hat{J}(t) + \int_t^T [\mathcal{D}\hat{J}(s) + \hat{J}_s(s)]ds \\ &+ \int_t^T [\sigma K(s)\hat{J}_K(s)]dB_s + \int_t^T [\hat{J}_z(s) - S(s)\hat{J}_K(s)]d\Phi_s \\ &+ \int_t^T [-\hat{J}_z(s) + S(s)\hat{J}_K(s)]d\Xi_s \end{aligned} \tag{85}$$

Arguments similar to the proofs of Lemmas 4 and 5 show that sufficient conditions for optimality of the controls are that

$$\begin{aligned} \mathcal{D}J^*(t) + J_t(t) + e^{-\rho t}u(c^*(t), z^*(t)) &= 0 \\ \int_t^T [J_z(t) - S(t)J_K(t)]d\Phi(t) &= 0 \\ \int_t^T [J_z(t) - S(t)J_K(t)]d\Xi(t) &= 0 \end{aligned}$$

We see that when

$$S(t) = \frac{J_z(t)}{J_K(t)}$$

the last two conditions are satisfied for any choice of $\Phi(t), \Xi(t) \in \mathbf{X}^+$. In particular, we can choose $\Xi(t) = 0$ and $\Phi(t) = \Phi^*(t)$ (i.e. the optimal policy in Lemma 5 which applies to the case of irreversible investment). Q.E.D.

The intuition behind the proof is that when $\omega(t) < \omega^*$, the marginal product of a unit of durable is lower than the marginal product of a unit of the nondurable (i.e. $J_z < J_K$). Hence it would be desirable to transfer a unit of durable into a unit of the nondurable. However if agents could only obtain $S(t) = J_z(t)/J_K(t) < 1$ units of nondurable for a unit of durable, they become indifferent between whether or not they do the transfer. Hence they would be happy to follow their exact policy in the case of irreversibility.

8.5 Proof of Theorem 3

The proof is available from the author upon request. Also see Hindy and Huang (1993). The present model with $A = 0$ corresponds exactly to their model. Hence their solution is directly applicable.

8.6 Proof of Theorem 4

The derivation of the lower bound was presented in the text. To find the upper bound, we observe that the value function of the constrained problem in (6) is dominated by the solution of the analogous determinstic constrained problem (i.e. with $\sigma = 0$). The proof for this claim is similar to the arguments used in Lemma 4 and Lemma 5. The main argument is that if a sufficiently nice solution to the deterministic problem exists, then it will satisfy the following differential inequality: $\max[J_t + (\mu K - c)J_K - \theta z J_z, J_z - J_K] = 0$. Using this and the fact that the deterministic value function is concave in K and in z, and in particular the fact that J_{KK} is negative, leads to the desired result. In a similar setting to the one in this paper, Shreve and Soner (1994) provide a proof that the value function is concave in Proposition 3.1, and that the deterministic value function dominates the stochastic one in Theorem 12.2.

An upper bound for the deterministic problem is computed as follows. Note that the dynamics of the capital stocks are given by

$$\begin{aligned} dK &= \mu K dt - c dt - d\Phi, \\ dz &= -\theta z dt + d\Phi, \end{aligned}$$

where $c > 0$ and $d\Phi > 0$. Given the current values of $K(t) > 0$ and $z(t) > 0$, we can solve for $z(s)$ at a time $s > t$ to find

$$z(s) = z(t)e^{-\theta(s-t)} + (\Phi(s) - \Phi(t)) - \theta \int_t^s (\Phi(u) - \Phi(t))e^{-\theta(s-u)} du.$$

We therefore conclude that

$$z(s) \leq z(t)\, e^{-\theta(s-t)} + K(t)\, e^{\mu(s-t)} < \Big(z(t) + K(t)\Big)\, e^{\mu(s-t)}.$$

Hence for any choice of $c(s)$ it must be the case that

$$e^{-\rho s} \frac{c(s)^A z(s)^B}{A+B} < \Big(z(t) + K(t)\Big)^B e^{-\mu B t} \times e^{-(\rho - \mu B)s} \frac{c(s)^A}{A+B}.$$

And in particular this holds for the $c^*(s)$ which was optimal in the deterministic constrained problem. Hence a bound for the value function is given by

$$\Big(z(t)+K(t)\Big)^B e^{-\mu Bt}\int_t^\infty e^{-(\rho-\mu B)s}\frac{c^*(s)^A}{A+B}ds.$$

By choosing among consumption processes feasible in the original problem we can show that

$$\sup_{c(\cdot)}\int_t^\infty e^{-(\rho-\mu B)s}\frac{c(s)^A}{A+B}ds = e^{-(\rho-\mu B)t}C^{1-A}\frac{K(t)^A}{A+B},$$

where $C=(A-1)/(A\mu+B\mu-\rho)$. Hence an upper bound for the deterministic, and hence the stochastic, constrained problem is given by

$$J_{high}(K,z,t)=e^{-\rho t}C^{1-A}\Big(z+K\Big)^B\frac{K^A}{A+B}.$$

Simple algebraic manipulation leads to the result in (47). Q.E.D.

8.7 Perturbation Analysis

Empirically Plausible Values of A and B

As has been discussed in Section 8.1, an empirically plausible range for δ is around 0.9 ± 0.2. A plausible range for γ is 2 ± 2. Hence the assumption that $B\equiv(1-\delta)(1-\gamma)$ is small seems justified.

The Perturbation Analysis

We will perturb the economy around $B=0$, using the standard only nondurable consumption economy as the base case. The value function of the representative agent's problem is described by equation (12), reproduced here for convenience

$$-(\theta(A+B)+\rho)g+(\theta+\mu)g'-(A-1)\left(\frac{g'}{Ae^\omega}\right)^{\frac{A}{A-1}}-\tfrac{1}{2}\sigma^2(g'-g'')=0$$

subject to the following three boundary conditions

$$\begin{aligned}(e^{-\omega^*}+1)g'(\omega^*)-(A+B)g(\omega^*) &= 0\\ ((A+B)e^{\omega^*}+1)g'(\omega^*)-(e^{\omega^*}+1)g''(\omega^*) &= 0\\ \lim_{\omega\to-\infty}g(\omega) &= 0\end{aligned}$$

Since the equation is of second order in the independent variable, one of these boundary conditions determines the no-investment region $(-\infty,\omega^*]$. We will obtain a closed form approximation to the equation by expanding around the solution for the case of no consumption flow from the durable (i.e. $B=0$). Let us define $\Omega\equiv K_t/z_t=e^{\omega(t)}$. Then we can write

$$\frac{1}{\Omega^*} = B\Omega_1+B^2\Omega_2+B^3\Omega_3+O(B^4) \tag{86}$$

$$g(\omega(t)) = g_0(\omega(t))+Bg_1(\omega(t))+B^2g_2(\omega(t))+O(B^3) \tag{87}$$

The no-trade region goes to $(-\infty,+\infty)$ as $B\to0$. We plug the expansion for the boundary of the no-investment region ω^* and for $g(\cdot)$ into the differential equation and the boundary conditions given above. The non-linear term in the above equation is handled by a Taylor series expansion around $B=0$, which linearizes this term for powers of B above 0 (despite this fact, the 0th order system has a well know solution (see Merton (1990)). That is

$$\left(\frac{g'}{Ae^{\omega}}\right)^{\frac{A}{A-1}} = \left(\frac{g_0'}{Ae^{\omega}}\right)^{\frac{A}{A-1}} + \frac{Be^{-\omega}}{A-1}\left(\frac{g_0'}{Ae^{\omega}}\right)^{\frac{1}{A-1}} g_1'(\omega) + \ldots$$

We collect terms of the same order in B. The solution to the 0th order system is given by

$$\begin{aligned} g_0(\omega(t)) &= C_0^{A-1} e^{\omega(t)A} \\ C_0 &= \frac{A-1}{A\mu - \rho + \frac{1}{2}(A-1)A\sigma^2} \end{aligned}$$

We note that all three boundary conditions of the 0th order system are satisfied by this solution.
The 1st order system has a solution given by

$$g_1(\omega) = C_{11}e^{A\omega} + C_{12}e^{k_1 A\omega} \tag{88}$$

$$C_{11} = -\theta C_0^{2-A} \tag{89}$$

$$C_{12} = \frac{C_0^{1-A}}{A(k_1-1)(1-A+Ak_1)} e^{A\omega^*(1-k_1)} \tag{90}$$

$$k_1 = \frac{-\theta + \frac{\mu-\rho}{A-1} + \frac{1}{2}(1+A)\sigma^2 + \sqrt{2(A\theta+\rho)\sigma^2 + (-\theta + \frac{\mu-\rho}{A-1} + \frac{1}{2}(1+A)\sigma^2)^2}}{A\sigma^2} \tag{91}$$

$$\Omega_1 = \frac{k_1 - 1}{1 - A + Ak_1} \tag{92}$$

The equation for k_1 has two solutions. One of them is dropped in order to satisfy the boundary condition for $g(-\infty)$ given in (14).
The 2nd order system is given by

$$g_2(\omega) = C_{21}e^{A\omega} + C_{22}e^{-A\omega+2Ak_1\omega} + C_{23}\omega e^{Ak_1\omega} + C_{24}e^{Ak_1\omega} \tag{93}$$

$$C_{21} = -\frac{(A-2)\theta^2 C_0^{3-A}}{(A-1)(-2C_0\rho + A^2C_0\sigma^2 + A(-2+2C_0\mu - C_0\sigma^2))} \tag{94}$$

$$C_{22} = \frac{C_0^{1-A}e^{-2A(k_1-1)\omega^*}k_1^2}{-2+2\theta C_0 + 2C_0\mu - C_0\sigma^2 - AC_0\sigma^2 + 3AC_0k_1\sigma^2} \times \frac{1}{(A-1)A^2(1-A+Ak_1)^2(k_1-1)^3} \tag{95}$$

$$C_{23} = -\frac{2\theta C_0^{2-A}e^{-A(k_1-1)\omega^*}}{(A-1)A(k_1-1)(-2+2\theta C_0 + 2C_0\mu - C_0\sigma^2 + 2AC_0k_1\sigma^2)} \tag{96}$$

$$\begin{aligned} C_{24} = {} & \frac{1}{A(1-A+Ak_1)^2(k_1-1)} \times \\ & \big[C_{12}(1+2A(k_1-1) - A^2(k_1-1)^3) + \\ & (1-A+Ak_1)\Big\{C_{11}e^{-A(k_1-1)\omega^*} - 2AC_{22}e^{A(k_1-1)\omega^*}(1+2A(k_1-1))(k_1-1) \\ & -C_{23}(1+A(k_1-1)(2+(1+A(k_1-1))\omega^*))\}\big] \end{aligned} \tag{97}$$

$$\begin{aligned} \Omega_2 = {} & \frac{C_0^{A-1}e^{A(k_1-2)\omega^*}(k_1-1)}{(1+Ak_1-A)^2} \times \\ & \Big(2AC_{22}e^{Ak_1\omega^*}(1+Ak_1-A))(k_1-1) + e^{A\omega^*}(C_{23}(1+Ak_1-A) - C_{12}k_1)\Big) \end{aligned} \tag{98}$$

Higher order approximations can also be computed, although the algebraic difficulty of doing so is readily apparent. The approximation presented here was derived with the use of symbolic computation software.

8.8 Proof of Lemma 1

From $S(\omega(t))$ given in (21) and the dynamics of $\omega(t)$ in (15), we can write the dynamics of $S(\omega(t))$ as follows

$$\begin{aligned}
dS_t &= \mu_S(\omega(t))dt + \sigma_S(\omega(t))dB_t + \phi_S(\omega(t))d\Phi_t \qquad (99)\\
\mu_S(\omega(t)) &= \frac{1}{2g'(\omega(t))^3}\Big\{2e^{\omega(t)}\mu_\omega(\omega(t))g'(\omega(t))\xi_S(\omega(t))\\
&\quad +\sigma^2[g'(\omega(t))^2((-1+2A+2B)g'(\omega(t)) - (A+B)g''(\omega(t)))\\
&\quad +(A+B)g(\omega(t))((g'(\omega(t)) - g''(\omega(t)))^2 + g''(\omega(t))^2 - g'(\omega(t))g^{(3)}(\omega(t)))]\Big\}\\
\sigma_S(\omega(t)) &= \frac{e^{\omega(t)}\sigma}{g'(\omega(t))^2}\,\xi_S(\omega(t))\\
\phi_S(\omega(t)) &= \frac{(1+e^{\omega(t)})}{z_t g'(\omega(t))^2}\,\xi_S(\omega(t))\\
\text{where }\ \xi_S(\omega(t)) &= (A+B-1)g'(\omega(t))^2 + (A+B)g(\omega(t))(g'(\omega(t)) - g''(\omega(t)))
\end{aligned}$$

Using the smooth-pasting and super-contact conditions in (10) and (13), we can show that $\xi_S(\omega^*) = 0$. Since $d\Phi_t = 0$ whenever $\omega(t) \neq \omega^*$ and $\phi_S(\omega^*) = 0$, we see that dS_t has no singular component. Also $\sigma_S(\omega^*) = 0$. Q.E.D.

8.9 Proof of Lemma 2

The consumption to nondurables ratio is given in (48). The expansion for the $g(\cdot)$ function is given in (87). The functions $g_0(\cdot)$, $g_1(\cdot)$, and $g_2(\cdot)$ are given in Section 8.7. We plug in the form of $g(\cdot)$ from (87) into (48) and do a Taylor series expansion around $B = 0$. The result in (49) follows. For convenience, k_1 is given by

$$k_1 = \frac{-\theta + \frac{\mu-\rho}{A-1} + \frac{1}{2}(1+A)\sigma^2 + \sqrt{2(A\theta+\rho)\sigma^2 + (-\theta + \frac{\mu-\rho}{A-1} + \frac{1}{2}(1+A)\sigma^2)^2}}{A\sigma^2}.$$

The derivative of k_1 with respect to ρ has the same sign as

$$\begin{aligned}
&\frac{\sqrt{2(A\theta+\rho)\sigma^2 + (-\theta + \frac{\mu-\rho}{A-1} + \frac{1}{2}(1+A)\sigma^2)^2} + (-\theta + \frac{\mu-\rho}{A-1} + \frac{1}{2}(1+A)\sigma^2) + \sigma^2(1-A)}{(1-A)A\sigma^2}\\
&= \frac{k_1}{1-A} + \frac{1}{A} = \frac{1+A(k_1-1)}{(1-A)\,A}.
\end{aligned}$$

[*Case of* $A > 0$]. Conditions (51) and (52) insure that for ρ equal to its minimum value of ρ_{min} in (51) we have $k_1 = 1$. We can then verify that the derivative of k_1 with respect to ρ is positive. Hence for $A > 0$, we have that $k_1 \geq 1$.

[*Case of* $A < 0$ *and* $A\theta + \rho > 0$]. First we observe that $k_1(\rho = -A\theta) = 0$ because $-\theta + \frac{\mu-\rho}{A-1} + \frac{1}{2}(1+A)\sigma^2 < 0$ when $\rho = -A\theta$ (this is easy to check by using condition (52)). If $\rho > -A\theta$, then it is easy to see that $k_1 < 0$.

[*Case of* $A < 0$ *and* $A\theta + \rho < 0$]. Consider the situation where $-A\theta > \rho > \rho_{min}$. Using conditions (51), (52), and (53) we can check that $k_1(\rho = \rho_{min}) = 1$. The sign of the derivative is therefore negative, and hence as ρ approaches $-A\theta$, k_1 decreases monotonically towards 0.

Hence we conclude that for $A < 0$, we have $k_1 \leq 1$. Also we see that for $\rho \in [\rho_{min}, -A\theta]$, $0 \leq k_1 \leq 1$. It remains to be shown that the interval $[\rho_{min}, -A\theta]$ is nonempty when $A < 0$. Using conditions (52) and (53) it is easy to check that $\rho_{min} < -A\theta$. Q.E.D.

8.10 Proof of Lemma 3

Recall from Theorem 2 that the implied relative risk aversion was given by

$$\gamma(\omega_s) = -(J_{KK}K)/J_K = 1 - g''(\omega(t))/g'(\omega(t)).$$

The second equality follows from the fact that $J(K, z, t) = e^{-\rho t}\frac{z^{A+B}}{A+B}g(\log(K/z))$. Plugging the form for $g(\cdot)$ developed in Section 8.7 into the above equation and doing a Taylor expansion around $B = 0$, we obtain the result in (57). Q.E.D.

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