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The Social Cost of Monopoly Power

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Abstract

A general equilibrium analysis of monopoly power is proposed as an alternative to the partial equilibrium analyses of monopolization common to most antitrust texts. This analysis introduces the notion of a cost minimizing market equilibrium. The empirical implications of this equilibrium concept for antitrust policy is derived in terms of a family of equilibrium inequalities over market data from observations on a market economy with competitive factor markets. The social cost of monopoly power is measured using Debreu's coefficient of resource utilization. That is, we propose Pareto optimality as the ultimate objective of antitrust policy.

Keywords: Monopoly power, Antitrust economics, Applied general equilibrium analysis

JEL Classification: D42, D58, D61, L12, L41

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In this paper we assume that the primary objective of antitrust law is promoting efficiency by controlling the creation, maintenance, and exercise of monopoly power by one or more firms. Monopoly power is difficult to measure, hence market share or the deviation between price and average variable cost are often suggested as operational proxies. In antitrust analyses of monopolization the major determinants of monopoly power are the marginal cost of production and the price elasticity of demand. We follow Debreu (1951) and measure the social cost of monopoly power by ρ, the coefficient of resource utilization. ρ is a measure of the allocative inefficiency due to monopolization and can be used to compute the money value of the associated real economic costs, i.e., deadweight loss.

We derive nonparametric estimates of the determinants of monopoly power and its social cost from a general equilibrium analysis of a history of observations on endogenous and exogenous market variables. In our model some firms are price-setters in their output markets, but all firms are assumed to be price-takers in competitive factor markets. The principal analytical constructs in our model are the notion of a "cost minimizing market equilibrium" and the implied family of equilibrium inequalities for a history of cost minimizing market equilibria.

In his classic 1957 expository paper on welfare economics, Bator gave a nonmathematical treatment of welfare maximization in general equilibrium theory. His general equilibrium model was the two-sector model with two factors of production, two consumption goods, two consumers or households and two producers or firms. Consumers have utility functions, endowments of the factors of production and limited liability shares of profits of each firm. Firms have production or cost functions and factors are inelastically supplied by consumers. Equilibrium in his model is the neoclassical notion of competitive equilibrium, where households and firms are price-takers. In equilibrium households maximize utility subject to their budget constraints and a consumer's income is the value of her endowment of factors at equilibrium prices plus her share of each firm's profits. Firms maximize profits by first choosing output such that its marginal cost is equal to the market price and then producing that output at minimum cost. In equilibrium all markets clear, i.e., supply is equal to demand in every market.

Brown and Heal (1983) used the same model to give an intuitive existence proof of two equilibrium notions occurring in public sector economics, i.e., average cost pricing equilibrium and marginal cost pricing equilibrium. In their model one firm has decreasing average cost and is regulated. One mode of regulation specifies the firm's output and requires the firm to produce that output at minimum cost and sell it at average cost. The Hotelling regulatory prescription specifies the firm's output, but requires the firm to produce that output at minimum cost and sell it at marginal cost. In this regime the firm is subsidized by taxes levied on consumers to cover the firm's losses. These regulated price-setting firms are price-takers in competitive factor markets and quantity-takers in their output markets.

This behavior is not characteristic of the price-setting firms in the static partial equilibrium models of monopoly pricing prevalent in monographs and case books on antitrust law, e.g., Fox et al. (2004), Gavil et al. (2002), Gellhorn and Kovacic

(1994), Hylton (2003), Morgan (2001), and Posner (2001). In this literature firms are assumed to face downward sloping demand curves and maximize profits by choosing the output where marginal revenue is equal to marginal cost and produce that output at minimum cost. The monopoly price is then determined by the firm's demand curve. It is now well established in the economic theory literature that the partial equilibrium model of monopoly pricing cannot be extended to the two-sector or, more generally, the Arrow–Debreu general equilibrium model. A profit maximizing firm that sets prices gives rise to a fundamental indeterminacy in general equilibrium models that determine relative prices, but not the absolute price level. In the general equilibrium theory literature this indeterminacy is called "the price normalization problem," i.e., equilibrium quantities will depend on the price normalization or equivalently the choice of numeraire. An illuminating discussion of the price normalization problem and its implications for economic policy can be found in Dierker and Grodal (1998).

Assuming competitive factor markets, a characteristic of competitive, regulated and monopolistic pricing of outputs in the three models discussed above is that in each instance the firm's output is produced at minimum cost. Since price-taking behavior in competitive factor markets is immune from the price normalization problem, we propose a new notion of market equilibrium, "cost minimizing market equilibrium." In this equilibrium concept households maximize utility subject to their budget constraints; factor markets are competitive; each firm produces its output at minimum cost; firms make nonnegative profits at the prevailing equilibrium prices; and all markets clear. In general, a cost minimizing market equilibrium is not Pareto optimal. See Appendix A for a formal discussion of cost minimizing market equilibria.

The testable implications of a history of observations on an economy, where in each observation the economy is in a cost minimizing equilibrium, is completely characterized by the equilibrium inequalities. This is a finite family of polynomial inequalities where the unknowns are utility levels and marginal utilities of income of consumers and the marginal costs of firms in each observation. The parameters in these inequalities are demands of households; each firm's demands for factors; the output of each firm; and the equilibrium prices of goods and factors in each observation.

The equilibrium inequalities exhaust the empirical content of our equilibrium notion in the sense that there exists utility functions for consumers and production or cost functions for firms such that in each observation the economy is in a cost minimizing equilibrium if and only if the equilibrium inequalities have a solution for the parameter values given by the observed market data. The equilibrium inequalities are derived in Appendix B.

In the antitrust analyses of monopoly pricing we previously cited, the economic costs of monopolization is measured in terms of social surplus, i.e., the sum of consumer and producer surplus. These texts, with the exception of Morgan (2001) use the Kaldor–Hicks notion of efficiency, based on potential compensation of losers by winners after a change in economic states, e.g., after the introduction of a tax. Social surplus is used to measure the potential compensation. Only in Posner (2001) do we find an explicit recognition that this measure of efficiency is "determined on the heroic assumption that a dollar is worth the same to everybody." In the jargon of economic theory consumer surplus only measures changes in consumers' welfare if the marginal utility of income is the same for every household, rich and poor alike. Social surplus as a measure of economic efficiency is basically a partial equilibrium notion, ill-suited for general equilibrium analysis.

In general equilibrium theory, the appropriate notion of efficiency or consumer welfare is Pareto optimality, called "allocative efficiency" in Morgan (2001). A state of the economy is said to be Pareto optimal if no consumer can be made better off by reallocating productive resources and engaging in mutually beneficial trades without making another consumer worse off. The first welfare theorem of general equilibrium theory states that every competitive equilibrium is Pareto optimal. Hence as in Posner (2001), "we value competition because it promotes efficiency, that is, as a means rather than as an end." If maximization of consumer welfare is the ultimate antitrust goal, then Pareto optimality and not competitive markets is the proper benchmark for measuring the economic costs of monopoly.

In antitrust analysis this cost is the difference between the monopoly output of a firm and the counterfactual competitive output of that firm. That is, the benchmark is competitive markets. Our analysis of this approach is in Appendix C.

The general equilibrium measure of allocative inefficiency due to monopolization introduced by Debreu (1951) is ρ, the coefficient of resource utilization. ρ is the smallest fraction of total resources capable of providing consumers with utility levels at least as great as those attained in the monopolized economic state. Hence the efficiency loss in real terms is $(1-\rho)\times$ total resources. That is, the economy can throw away $(1-\rho)\times$ total resources and not make anyone worse off. In fact, relative to this reduced resource endowment, the original monopoly state is Pareto optimal.

The second welfare theorem of general equilibrium theory states that every Pareto optimal economic state can be realized as a competitive equilibrium with lump sum transfers of income between households. In this sense the basic intuition of antitrust analysis is correct that competitive markets serve as a benchmark for measuring monopoly power. In particular, we agree with Hylton (2003) that measuring monopoly power with the Lerner index overstates the monopoly surcharge, if we drop the unrealistic assumption that the monopolist has constant marginal costs. In the partial equilibrium models of monopoly pricing used in antitrust analysis, the "correct" price-cost margin is the difference between the observed monopoly price and the counterfactual price of the competitive output.

We have argued that the partial equilibrium analysis of monopoly pricing found in the previously cited antitrust texts is seriously flawed. It relies on untenable assumptions such as equal marginal utilities of income for all consumers. Moreover, it ignores the recently proven impossibility of extending the paradigmatic general equilibrium model of Arrow and Debreu, Nobel Laureates in economics, to include price-setting profit-maximizing firms.

As an alternative mode of analysis, not subject to these criticisms, we suggest general equilibrium analysis where the equilibrium concept is "cost minimizing market equilibria." In our approach, consistent with the informal welfare analysis appearing in antitrust case books such as Morgan (2001), we propose Pareto optimality as the ultimate objective of antitrust policy. A natural measure of the deviation of an economic state from Pareto optimality is Debreu's coefficient of resource utilization, ρ. The economic costs of monopolization are easily computed from ρ. See Appendix D for an illustrative example, using the two-sector model. In practice ρ must be estimated from market data. Given a history of observations on household and firm behavior in a market economy with competitive factor markets, the equilibrium inequalities for cost minimizing market equilibria can be used to derive upper and lower bounds on ρ in each observation, hence bounding the deadweight loss in each observation. Proposed remedies can also be evaluated in terms of ρ, i.e., if ρ_i is the value of ρ for remedy i, where $i = 1, 2, ..., T$. Then the remedy with the smallest deadweight loss is the remedy with the largest coefficient of resource utilization. In Appendix E we compute ρ in an example using the two-sector model and simulated market data.

The table below summarizes the principal differences between the partial and general equilibrium analyses of monopoly power.

	Partial equilibrium	General equilibrium
Firms' behavior	Price-setting Profit maximizer	Cost minimizing market equilibrium
Benchmark behavior	Competitive market	Pareto optimal economic states
Measure of economic cost	Social surplus	Coefficient of resource utilization, ρ
Computation of deadweight loss	Varian's cost minimizing inequalities	Equilibrium inequalities
Remedy	Increase competition	Increases ρ

We illustrate these differences in Figures 1a and 1b. Figure 1b is discussed in Appendix D.

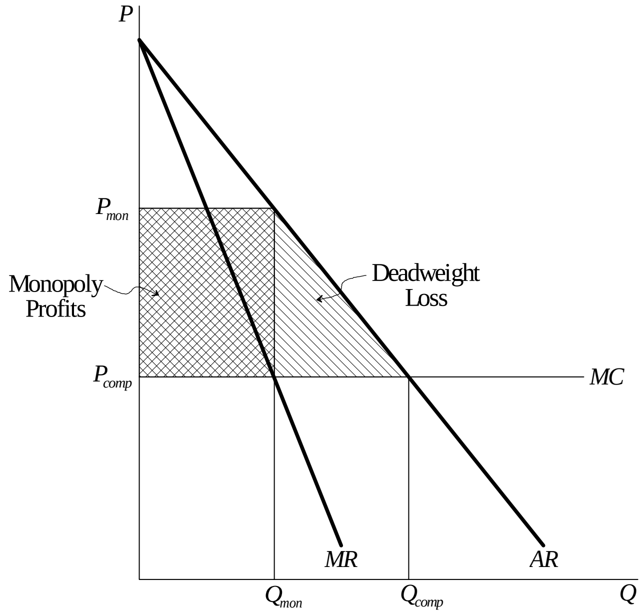


Figure 1a. Partial Equilibrium Analysis

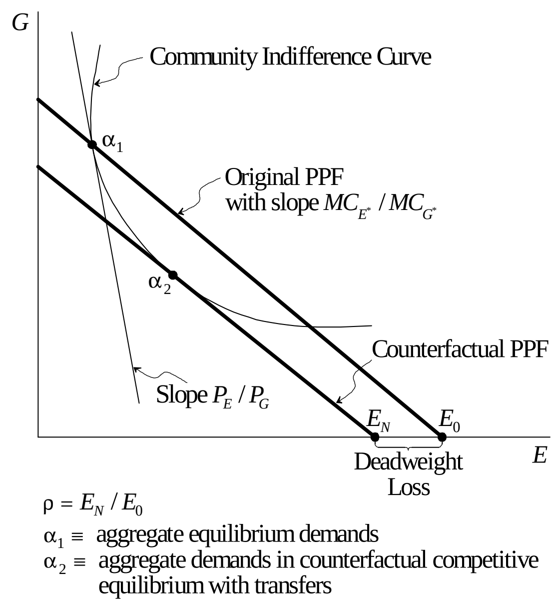


Figure 1b. General Equilibrium Analysis

The economic model used in the appendices is the two-sector general equilibrium model. The intended interpretation is a model of two product markets where the monopoly good and the competitive good are close substitutes in consumption. The affect of other products on the welfare of consumers can be explicitly modeled by assuming households have quasilinear utility functions, linear in a third composite commodity. This is common practice in the I.O. literature.

The Justice Department's Merger Guidelines suggest constraints on monopoly pricing, arising from close substitutes in consumption, as a means of defining the relevant market for abuse of monopoly power or the efficacy of a proposed merger. If the relevant market for analysis contains several products, then we propose the multisector general equilibrium model and the associated equilibrium inequalities for cost minimizing market equilibria as a methodology for estimating the social cost of monopolization.

This is a paper on antitrust analysis from a general equilibrium perspective, but we close with a brief comment on the empirical theory of monopolistic competition and the welfare implications of market power. Using the methodology developed in this paper you can estimate the social cost of market power in a multisector general equilibrium model of monopolistic competition, where in the cost minimizing market equilibria the market price of each good exceeds its marginal cost of production, i.e., every firm has some market power. In this model the number of firms is fixed, i.e., no entry or exit; goods or products are differentiated; and households may have heterogeneous tastes. The measure of consumer welfare or the social cost of market power is, again, Debreu's coefficient of resource utilization.

Appendix A: Cost Minimizing Market Equilibrium

In this appendix we give a formal definition of a cost minimizing market equilibrium in the two-sector general equilibrium model. The characterization of cost minimization in Varian (1984) plays an essential role in our analysis. For readers unfamiliar with the geometry of the two-sector model, we recommend the paper by Bator (1957). Here we follow the notation in Brown and Heal (1983).

The inputs or factors are capital (K) and labor (L). The outputs or goods are natural gas (G) and electricity (E). Each household has a utility function denoted U_x and U_y. Endowments and shareholdings in firms are given by (K_x, L_x), (K_y, L_y); $(\theta_{XG}, \theta_{XE})$, $(\theta_{YG}, \theta_{YE})$. Each firm has a production function, F_G and F_E, or equivalently, cost functions C_G and C_E. Let $K = K_x + K_y$ and $L = L_x + L_y$.

We make the same assumptions regarding firms and households as Bator, with one exception: we do not assume constant returns to scale in both firms, but firms are assumed to exhibit diminishing marginal rate of substitution along any isoquant, that is, the market for factors is competitive. Under these assumptions, we construct the Edgeworth–Bowley box for production and the social production possibility frontier, PPF.

Let P_G and P_E denote the prices of natural gas and electricity, and w and r denote the prices of labor and capital. The marginal rate of transformation (MRT) at a point $(\tilde{G}, \tilde{E})$ on the PPF is simply the absolute value of the slope of the frontier at that point and will be denoted $P_{\tilde{E}}/P_{\tilde{G}}$. A point $(\tilde{G}, \tilde{E})$ is said to be production efficient if it lies on the PPF.

Each point $(\tilde{G}, \tilde{E})$ on the PPF determines a unique point in the Edgeworth–Bowley box for production, that is, the point on the efficiency locus corresponding to the tangency of the isoquants defined by $F_E(L_E, K_E) = \tilde{E}$ and $F_G(L_G, K_G) = \tilde{G}$. The slope of their common tangent line at this point will be denoted as w/r and is the marginal rate of technical substitution (MRTS) at this point.

We shall use repeatedly that the MRT at a point $(\tilde{G}, \tilde{E})$ is the ratio of the marginal costs; that is $P_{\tilde{E}}/P_{\tilde{G}} = \frac{\partial C_E(w/r,\tilde{E})/\partial E}{\partial C_G(w/r),\tilde{G})/\partial G}$.

A consumer's demand for goods derive from utility maximization subject to her budget constraint:

$$\frac{\partial U_i/\partial E_i}{\partial U_i/\partial G_i} = \frac{P_E}{P_G} \tag{1}$$

$$P_E E_i + P_G G_i = I_i = wL_i + rK_i + \theta_{iG}(P_G G - wL_G - rK_G) + \theta_{iE}(P_E E - wL_E - rK_E), \text{ for } i = x, y. \tag{2}$$

A firm's demand for factors derive from cost minimization subject to its output constraint:

$$\frac{\partial F_j/\partial L_j}{\partial F_j/\partial K_j} = \frac{w}{r} \tag{3}$$

$$F_j(L_j, K_j) = j, \text{ for } j = E, G. \tag{4}$$

A cost minimizing equilibrium is defined as a set of relative prices P_E/w, P_G/w and r/w; consumer's demands for goods E_x, G_x and E_y, G_y; firm's demands for factors

L_E, K_E and L_G, K_G; and output levels E and G such that all markets clear. That is,
Product Markets:

$$E_x + E_y = E \tag{5}$$

$$G_x + G_y = G \tag{6}$$

Factor Markets:

$$L_E + K_G = L \tag{7}$$

$$K_E + K_G = K \tag{8}$$

and firms make nonnegative profits:

$$P_E E \geq wL_E + rK_E \tag{9}$$

$$P_G G \geq wL_G + rK_G. \tag{10}$$

Because of Walras' Law, equation (8) is redundant. This model is indeterminate in the sense that there are fewer equations (11) than unknowns (13). Despite the indeterminacy, this system of equations and inequalities suggests a family of equilibrium inequalities for a history of market data that allows us to infer the exercise of monopoly power.

Appendix B: Equilibrium Inequalities

The Afriat inequalities consist of a finite number of polynomial inequalities derived from a finite number of observations on a consumer's demands: $x_1, x_2, ..., x_n$ at market prices: $p_1, p_2, ..., p_n$. For each pair of observations, i and j, there is a pair of inequalities

$$\begin{aligned} V_i &\leq V_j + \lambda_j p_j \cdot (x_i - x_j) \text{ and} \\ V_j &\leq V_i + \lambda_i p_i \cdot (x_j - x_i), \end{aligned}$$

where V_i is the utility level and λ_i is the marginal utility of income in observation i. A utility function U is said to rationalize the data $\{(p_1, x_1), ..., (p_n, x_n)\}$ if for all i, $U(x_i) \geq U(x)$ for all x such that $p_i \cdot x \leq p_i \cdot x_i$. Afriat's celebrated theorem is that the data is rationalized by a concave, monotonic and continuous utility function U if and only if the Afriat inequalities are solvable. Moreover, a rationalizing U can be constructed from each solution of the Afriat inequalities. See Afriat (1967) and Varian (1982) for further discussion.

Varian's cost minimizing inequalities consist of a finite number of polynomial inequalities derived from a finite number of observations on a firm's outputs: $f_1, f_2, ..., f_n$; factor demands: $y_1, y_2, ..., y_n$; and factor prices: $q_1, q_2, ..., q_n$. For each pair of observations, i and j, there is a pair of inequalities:

$$\begin{aligned} f_i &\leq f_j + \beta_j q_j \cdot (y_i - y_j) \text{ and} \\ f_j &\leq f_i + \beta_i q_i \cdot (y_j - y_i), \end{aligned}$$

where β_i is the reciprocal of the marginal cost in observation i. A production function f is said to rationalize the data if for all i, $f(y_i) = f_i$ and $f(y) \geq f(y_i)$ implies $q_i^i \cdot y \geq q_i^i \cdot y^i$. That is, y_i minimizes the cost over all bundles of factors that can produce at least f_i. Varian (1984) proves the important result that the cost minimizing inequalities are solvable if and only if there exists a continuous monotonic quasiconcave, i.e., diminishing marginal rate of substitution along any isoquant, function that rationalizes the data.

Brown and Matzkin (1996) introduced the Walrasian equilibrium inequalities as a means of testing the Walrasian model of a competitive economy with market data. Here we propose an analogous family of polynomial inequalities that characterize a history of cost minimizing market equilibria in an economy with competitive factor markets. The equilibrium inequalities consists of: the Afriat inequalities for each consumer; her budget constraint in each observation; the cost minimizing inequalities for each firm; the market clearing conditions for the goods and factor markets in each observation; and the nonnegative profit conditions for each firm in each observation. An explicit example for the two-sector model is given in Appendix E.

Appendix C: The Major Determinants of Market Power

In this appendix we give nonparametric estimates of the marginal cost and price elasticity of demand of a firm minimizing the cost of production in a model where all firms are price-takers in competitive factor markets. It suffices to consider the firm producing electricity in the two-sector general equilibrium model. Suppose this firm is alleged to have abused its market power in the previous year — think of California before the recall election of Gray Davis. During the year we observe that on three occasions the firm produced different outputs at the prevailing market prices. We observe the outputs, factor demands, and factor prices on each occasion. We do not know the firm's cost or production function, but if the firm is producing its output at minimum costs in competitive factor markets, then the following figure must hold for the unobserved isoquants of the production function.

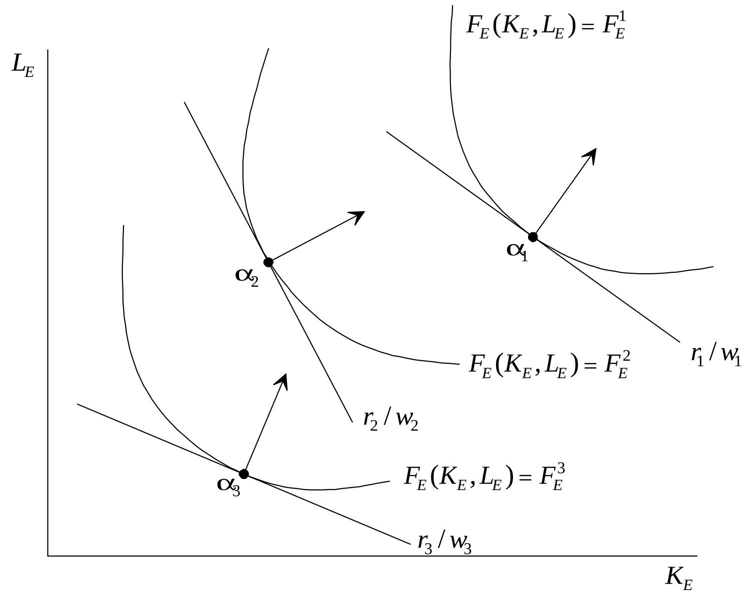


Figure 2

We have three isoquants where $F_E^1 > F_E^2 > F_E^3$. The slopes or factor price ratios are: r_1/w_1, r_2/w_2, and r_3/w_3. The factor demands are: (K_E^1, L_E^1), (K_E^2, L_E^2), and (K_E^3, L_E^3), denoted α^1, α^2, and α^3 in the figure.

Using Varian's cost minimizing inequalities, we have:

$$F_E^3 < F_E^2 + \lambda_2[(w_2 L_E^3 + r_2 K_E^3) - (w_2 L_E^2 + r_2 K_E^2)] \tag{11}$$

$$F_E^1 < F_E^2 + \lambda_2[(w_2 L_E^1 + r_2 K_E^1) - (w_2 L_E^2 + r_2 K_E^2)] \tag{12}$$

where λ_2 is the reciprocal of the marginal cost, MC_E^2, of producing F_E^2. solving for MC_E^2, we obtain:

$$MC_E^2 > \frac{(w_2 L_E^2 + r_2 K_E^2) - (w_2 L_E^3 + r_2 K_E^3)}{F_E^2 - F_E^3} \equiv B_{\min}^2 \tag{13}$$

$$MC_E^2 < \frac{(w_2 L_E^1 + r_2 K_E^1) - (w_2 L_E^2 + r_2 K_E^2)}{F_E^1 - F_E^2} \equiv B_{\max}^2 \tag{14}$$

The right-hand side of (13) may be negative, as seen in the next figure, and hence uninformative. We compute these bounds in an example in Appendix E.

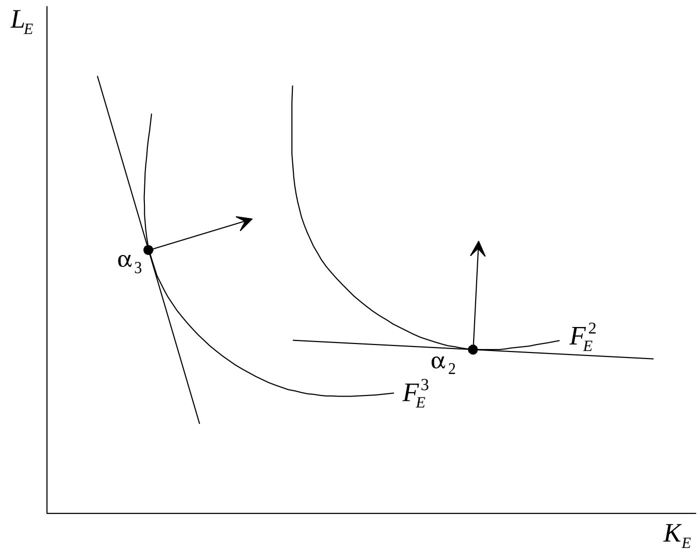


Figure 3

Assuming that the firm is making nonnegative profits and the right-hand side of (13) is positive, we can derive an upper bound on monopoly power, measured by the price elasticity of demand for electricity, ε_E. Nonnegative profits imply that, the marginal revenue, $MR_E^2 > MC_E^2$. Hence

$$\frac{P_E^2 - B_{\min}^2}{P_E^2} > \frac{P_E^2 - MC_E^2}{P_E^2} > -\frac{1}{\varepsilon_E^2}. \tag{15}$$

If one makes the additional assumptions of a linear demand curve for electricity, constant marginal utilities of income for consumers and constant marginal cost. Then we can derive a lower bound on the deadweight loss as measured by social surplus. There are dubious and nontrivial assumptions. Of course, we also can estimate a lower bound on the monopoly's economic profits and Posner's measure of the social cost of monopolization, i.e., monopoly profits + deadweight loss.

Appendix D: The Coefficient of Resource Utilization

A common argument made by firms seeking to merge is that the merger will result in efficiencies, e.g., a reduction in the labor force or reduction in capital utilization, say fewer plants. Put simply the merged firm will produce more with less labor or capital. If true then the original economic state was not Pareto optimal. In a similar vein the intuition that competition and free entry to markets enhances "efficiency" follows from the belief that firms with lower marginal costs will enter profitable markets and produce the market output at lower cost. If true then the original economic state was not Pareto optimal. The common feature of these two examples is that in each case the same level of consumer satisfaction can be realized with fewer total resources. These wasted resources constitute the opportunity cost of the inefficiencies in the original economic state. Debreu's coefficient of resource utilization, ρ, is a quantitative measure of these inefficiencies or real opportunity costs.

We illustrate the computation of ρ with the two-sector model. Suppose the given economic state of the model is a cost minimizing market equilibrium where $P_E/P_G \neq MC_E/MC_G$. As shown in Bator (1957) this violates one of the necessary conditions for Pareto optimality in the two-sector model.

Suppose in equilibrium household x consumes (E_x, G_x) and household y consumes (E_y, G_y). ρ is the minimum α between 0 and 1 where the given two-sector model with reduced social endowments αK and αL can produce sufficient electricity $\bar{E}$ and natural gas $\bar{G}$ such that:

$$U_x(\bar{E}_x, \bar{G}_x) \geq U_x(E_x, G_x) \tag{16}$$

$$U_y(\bar{E}_y, \bar{G}_y) \geq U_y(E_y, G_y) \tag{17}$$

$$\bar{E}_x + \bar{E}_y = \bar{E} \tag{18}$$

$$\bar{G}_x + \bar{G}_y = \bar{G} \tag{19}$$

$$\bar{E} = F_E(\bar{L}_E, \bar{K}_E) \tag{20}$$

$$\bar{G} = F_G(\bar{L}_G, \bar{K}_G) \tag{21}$$

$$\bar{L}_E + \bar{L}_G = \alpha L \tag{22}$$

$$\bar{K}_E + \bar{K}_G = \alpha K \tag{23}$$

We denote the Lagrange multipliers for constraints (22) and (23) in the constrained minimization problem defining ρ as $\bar{w}$ and $\bar{r}$. These "shadow prices" as they are called by economists are used by Debreu to give an intrinsic valuation of the economic costs of inefficiency. He defines the opportunity or economic cost in real terms as the vector $\langle (1-\rho)K, (1-\rho)L \rangle$ and the deadweight loss as $(1-\rho)[\bar{w}L + \bar{r}K]$.

In practice, ρ must be estimated from market data. Here we use the equilibrium inequalities. Given a history of observations on the two-sector model, the equilibrium inequalities are solvable linear inequalities in the utility levels and marginal utilities of households and the marginal costs of firms, for parameter values given by the observed market data, if and only if this is a history of cost minimizing market equilibria. Each solution determines a utility function for each household and a production function for each firm that rationalizes the market data in each observation. Hence for a given observation and a given solution of the equilibrium inequalities we can solve the minimization problem for ρ defined by equations (16)–(23). Keeping the observation fixed we compute the minimum ρ, $\rho_{\min}$, and the maximum ρ, $\rho_{\max}$, over the set of solutions to the equilibrium inequalities. Hence the "true" ρ is in the interval $[\rho_{\min}, \rho_{\max}]$.

A clever illustration of the coefficient of resource allocation was suggested by T.N. Srinivasan. Suppose both firms in the two-sector model have constant returns to scale. Then each firm has constant marginal cost. Using the fact that the MRT is the ratio of the marginal costs at each point on the PPF, we see that the PPF is a straight line. The outputs (E^*, G^*) produced in a cost minimizing market equilibrium lie on the PPF, as a consequence of competitive factor markets and production at minimum cost. Finally the community indifference curve passing through the point (E^*, G^*) is the boundary of the set of points $(\hat{E}, \hat{G})$ where

$$U_x(\hat{E}_x, \hat{G}_x) \geq U_x(E_x^*, G_x^*) \tag{24}$$

$$U_y(\hat{E}_y, \hat{G}_y) \geq U_y(E_y^*, G_y^*) \tag{25}$$

$$\hat{E}_x + \hat{E}_y = \hat{E} \tag{26}$$

$$\hat{G}_x + \hat{G}_y = \hat{G} \tag{27}$$

$$E_x^* + E_y^* = E^* \tag{28}$$

$$G_x^* + G_y^* = G^* \tag{29}$$

Hence we have the following figure:

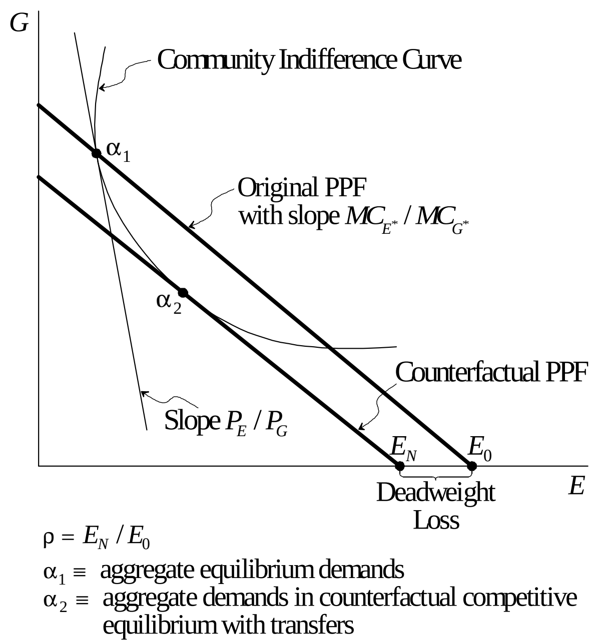


Figure 4

α_1 is the output (E^*, G^*) produced in the cost minimizing market equilibrium. $\alpha_2 = (\hat{E}, \hat{G})$ and satisfies (24)–(29). The social endowments used to produce α_2 are ρK and ρL, where K and L are the original social endowments of capital and labor. If the slope of the PPF is P_E^*/P_G^*, then the deadweight loss is $P_E^*(E^* - \hat{E}) + P_G^*(G^* - \hat{G})$, and ρ is the ratio E_N/E_0.

Appendix E: Numerical Examples

The first task is simulating market data. To this end we specify factor endowments for households in a two-sector general equilibrium model and solve the equilibrium inequalities for three periods. In each period the unknowns are utility levels, marginal utilities of income and demands of households, output levels, marginal costs of production and factor demands of firms, and prices of goods and factors in each period. We use NMaximize to solve this family of polynomial inequalities.

NMaximize is an algorithm in Wolfram (2003) for computing local maxima in polynomial programs, i.e., constrained maximization problems where the objective function and constraints are multivariate polynomials. By setting the objective function equal to zero, NMaximize is an effective algorithm for deciding if a family of polynomial inequalities is solvable. That is, if the given system is inputed as constraints into NMaximize where the objective function is a constant, say zero, then the output will be a solution to the inequalities or the announcement that the system is inconsistent. If the range of the objective function in a polynomial program is known and bounded, then a binary line search on the range, using NMaximize, will produce the global maximum.

To simplify the simulation's coding, we departed from the notation in Brown and Heal (1983). In our simulation the utility levels of households are denoted as U_i and V_i, the demands of households as x_{ij} and y_{ij}; the output levels of firms as F_i and G_i; the factor demands of firms as w_{ij} and z_{ij}; the prices of goods as p_{ij}; the prices of factors as q_{ij}.

In this notation, i denotes the period and j the good or factor. The $j = 1$ good, electricity, is produced by the F-firm. The $j = 1$ factor, labor, is the numeraire. Hence, $q_{11} = q_{21} = 1$. The G-firm has a homogenous production function and the F-firm has a concave production function, i.e., increasing average costs. The marginal cost of the F-firm in period i is denoted as $1/M_i$. The cost minimizing inequalities for these technologies, as they appear in the simulation, are taken from Varian (1989).

The Afriat inequalities for homogenous utility functions are taken from Varian (1982).

Here is the Mathematica code for the market data simulation and the results of one run:

$\{U_1 > 1,\ U_2 > 1,\ U_3 > 1,\ x_{11} > 0,\ x_{12} > 0,\ x_{21} > 0,\ x_{22} > 0,\ x_{31} > 0,\ x_{32} > 0,\ V_1 > 1,\ V_2 > 1,$
$V_3 > 1,\ y_{11} > 0,\ y_{12} > 0,\ y_{21} > 0,\ y_{22} > 0,\ y_{31} > 0,\ y_{32} > 0,\ F_1 > 0,\ F_2 > 0,\ F_3 > 0,\ w_{11} > 0,$
$w_{12} > 0,\ w_{21} > 0,\ w_{22} > 0,\ w_{31} > 0,\ w_{32} > 0,\ G_1 > 0,\ G_2 > 0,\ G_3 > 0,\ z_{11} > 0,\ z_{12} > 0,\ z_{21} > 0,$
$z_{22} > 0,\ z_{31} > 0,\ z_{32} > 0,\ p_{11} > 0,\ p_{12} > 0,\ p_{21} > 0,\ p_{22} > 0,\ p_{31} > 0,\ p_{32} > 0,\ q_{12} > 0,$
$q_{22} > 0,\ q_{32} > 0,\ M_1 > 0,\ M_2 > 0,\ U_3 * \{p_{21},\ p_{22}\}.\{x_{21}, x_{22}\} \le U_2 * \{p_{21}, p_{22}\}.\{x_{31}, x_{32}\},$
$U_2 * \{p_{31}, p_{32}\}.\{x_{31}, x_{32}\} \le U_3 * \{p_{31}, p_{32}\}.\{x_{21}, x_{22}\},$
$U_2 * \{p_{11}, p_{12}\}.\{x_{11}, x_{12}\} \le U_1 * \{p_{11}, p_{12}\}.\{x_{21}, x_{22}\},$
$U_1 * \{p_{21}, p_{22}\}.\{x_{21}, x_{22}\} \le U_2 * \{p_{21}, p_{22}\}.\{x_{11}, x_{12}\},\ \{p_{31}, p_{32}\}.\{x_{31}, x_{32}\} = \{1, q_{32}\}.\{120, 0\},$
$\{p_{21}, p_{22}\}.\{x_{21}, x_{22}\} = \{1, q_{22}\}.\{90, 0\},\ \{p_{11}, p_{12}\}.\{x_{11}, x_{12}\} = \{1, q_{12}\}.\{60, 0\},$
$\{p_{21}, p_{22}\}.\{x_{21}, x_{22}\} = \{1, q_{22}\}.\{90, 0\},$

$V_3 * \{p_{21}, p_{22}\}.\{y_{21}, y_{22}\} \leq V_2 * \{p_{21}, p_{22}\}.\{y_{31}, y_{32}\},$
$V_2 * \{p_{31}, p_{32}\}.\{y_{31}, y_{32}\} \leq V_3 * \{p_{31}, p_{32}\}.\{y_{21}, y_{22}\},$
$V_2 * \{p_{11}, p_{12}\}.\{y_{11}, y_{12}\} \leq V_1 * \{p_{11}, p_{12}\}.\{y_{21}, y_{22}\},$
$V_1 * \{p_{21}, p_{22}\}.\{y_{21}, y_{22}\} \leq V_2 * \{p_{21}, p_{22}\}.\{y_{11}, y_{12}\},\ \{p_{31}, p_{32}\}.\{y_{31}, y_{32}\}$
$= \{1, q_{32}\}.\{0, 75\} + p_{31} * F_3 - \{1, q_{32}\}.\{w_{31}, w_{32}\},$
$\{p_{21}, p_{22}\}.\{y_{21}, y_{22}\} = \{1, q_{22}\}.\{0, 50\} + p_{21} * F_2 - \{1, q_{22}\}.\{w_{21}, w_{22}\},\ \{p_{11}, p_{12}\}.\{y_{11}, y_{12}\}$
$= \{1, q_{12}\}.\{0, 25\} + p_{11} * F_1 - \{1, q_{12}\}.\{w_{11}, w_{12}\},\ p_{21} * F_2 - \{1, q_{22}\}.\{w_{21}, w_{22}\} \geq 0,$
$\{p_{21}, p_{22}\}.\{y_{21}, y_{22}\} = \{1, q_{22}\}.\{0, 50\} + p_{21} * F_2 - \{1, q_{22}\}.\{w_{21}, w_{22}\},$
$F_2 \leq F_3 + M_3 * \{1, q_{32}\}.(\{w_{21}, w_{22}\} - \{w_{31}, w_{32}\}),\ F_3 \leq F_2 + M_2 * \{1, q_{22}\}.(\{w_{31}, w_{32}\} - \{w_{21}, w_{22}\}),$
$F_1 \leq F_2 + M_2 * \{1, q_{22}\}.(\{w_{11}, w_{12} < -\{w_{21}, w_{22}\}),\ F_2 \leq F_1 + M_1 * \{1, q_{12}\}.(\{w_{21}, w_{22}\}$
$- \{w_{11}, w_{12}\}),$
$M_1 * \{1, q_{12}\}.\{w_{11}, w_{12}\} < F_1,\ M_2 * \{1, q_{22}\}.\{w_{21}, w_{22}\} < F_2,\ M_3 * \{1, q_{32}\}.\{w_{31}, w_{32}\} < F_3,$
$p_{21} * F_2 - \{1, q_{22}\}.\{w_{21}, w_{22}\} \geq 0,\ p_{31} * F_3 - \{1, q_{32}\}.\{w_{31}, w_{32}\} \geq 0,$
$p_{11} * F_1 - \{1, q_{12}\}.\{w_{11}, w_{12}\} = 0,\ \{1, q_{32}\}.\{z_{21}, z_{22}\} * G_3 \leq \{1,\ q_{32}\}.\{z_{31}, z_{32}\} * G_2,$
$\{1, q_{22}\}.\{z_{31}, z_{32}\} * G_2 \leq \{1, q_{22}\}.\{z_{21}, z_{22}\} * G_3,$
$\{1, q_{22}\}.\{z_{21}, z_{22}\} * G_1 \leq \{1, q_{22}\}.\{z_{11}, z_{12}\} * G_2,$
$\{1, q_{12}\}.\{z_{11}, z_{12}\} * G_2 \leq \{1, q_{12}\}.\{z_{21}, z_{22}\} * G_1,\ w_{31} + z_{31} = 120,\ w_{32} + z_{32} = 75,\ w_{21} + z_{21} = 90,$
$w_{22} + z_{22} = 50,\ w_{11} + z_{11} = 60,\ w_{12} + z_{12} = 25,\ w_{21} + z_{21} = 90,\ w_{22} + z_{22} = 50,\ x_{11} + y_{11} = F_1,$
$x_{12} + y_{12} = GL,\ x_{21} + y_{21} = F_2,\ x_{22} + y_{22} = G_2,\ x_{31} + y_{31} = F_3,\ x_{32} + y_{32} = G_3,\ M_3 > 0\}$

$\{F_3 \to 0.12086015700510068,\ G_3 \to 0.02113212603605529,\ M_3 \to 0.05712764322360773,$
$p_{31} \to 35.07799741549837,\ p_{32} \to 5678.541808898278,\ q_{32} \to 1.206568950723324^{-29},$
$U_3 \to 2.9631551816073056,\ V_3 \to 1.0537259131402723,\ w_{31} \to 0.00033879335219921813,$
$w_{32} \to 74.99408293177184,\ x_{31} \to 9.684934322017799^{-6},\ x_{32} \to 0.02113212587144456,$
$y_{31} \to 0.12085047207077866,\ y_{32} \to 1.646107289520379^{-10},\ z_{31} \to 119.9996612066478,$
$z_{32} \to 0.005917068228162191,\ F_1 \to 1.408211590367411,\ F_2 \to 1.365439889409406,$
$G_1 \to 0.0020946023279088287,\ G_2 \to 0.011633540777631766,\ M_1 \to 2.0437503246274403^{-19},$
$M_2 \to 0.05425525266887062,\ p_{11} \to 37.175585332603234,\ p_{12} \to 6686.675930253387,$
$p_{21} \to 54.16966546761761,\ p_{22} \to 8769.163943093063,\ q_{12} \to 0.2540209672621788,$
$q_{22} \to 0.7228466215440974,\ U_1 \to 1.3132899342122344,\ U_2 \to 1.439111016814532,$
$V_1 \to 1.4034789404061154,\ V_2 \to 13.39800717731972,\ w_{11} \to 49.65901568759563,$
$w_{12} \to 10.572282884453157,\ w_{21} \to 23.939580380550325,\ w_{22} \to 0.25776475034942337,$
$x_{11} \to 1.326713126043797,\ x_{12} \to 0.0015970062079065607,\ x_{21} \to 0.6893762026321762,$
$x_{22} \to 0.006004759640005697,\ y_{11} \to 0.08149846432361406,\ y_{12} \to 0.0004975961200022679,$
$y_{21} \to 0.6760636867772296,\ y_{22} \to 0.005628781137626068,\ z_{11} \to 10.340984312404375,$
$z_{12} \to 14.427717115546843,\ z_{21} \to 66.06041961944968,\ z_{22} \to 49.74223524965058\}$

{True, True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True}

Given the simulated values of F_1, F_2, the associated factor demands and factor prices in each period, we compute a upper bound, $B^2_{\max}$, on the marginal cost of producing F_2. This bound is given by equation (14). An upper bound on the elasticity of demand is an immediate consequence. Here are those results:

$\{\{1, q_{32}\}.(\{w_{21}, w_{22}\} - \{w_{31}, w_{32}\})\}/.$
$\{F_3 \to 0.12086015700510068,\ G_3 \to 0.02113212603605529,\ M_3 \to 0.05712764322360773,$
$p_{31} \to 35.07799741549837,\ p_{32} \to 5678.541808898278,\ q_{32} \to 1.206568950723324^{-29},$
$U_3 \to 2.9631551816073056,\ V_3 \to 1.0537259131402723,\ w_{31} \to 0.00033879335219921813,$
$w_{32} \to 74.99408293177184,\ x_{31} \to 9.684934322017799^{-6},\ x_{32} \to 0.02113212587144456,$
$y_{31} \to 0.12085047207077866,\ y_{32} \to 1.646107289520379^{-10},\ z_{31} \to 119.9996612066478,$
$z_{32} \to 0.005917068228162191,\ F_1 \to 1.408211590367411,\ F_2 \to 1.365439889409406,$
$G_1 \to 0.0020946023279088287,\ G_2 \to 0.011633540777631766,\ M_1 \to 2.0437503246274403^{-19},$
$M_2 \to 0.05425525266887062,\ p_{11} \to 37.175585332603234,\ p12 \to 6686.675930253387,$
$p_{21} \to 54.16966546761761,\ p_{22} \to 8769.163943093063,\ q_{12} \to 0.2540209672621788,$
$q_{22} \to 0.7228466215440974,\ U_1 \to 1.3132899342122344,\ U_2 \to 1.439111016814532,$
$V_1 \to 1.4034789404061154,\ V_2 \to 13.839800717731972,\ w_{11} \to 49.65901568759563,$
$w_{12} \to 10.572282884453157,\ w_{21} \to 23.939580380550325,\ w_{22} \to 0.25776475034942337,$
$x_{11} \to 1.326713126043797,\ x_{12} \to 0.0015970062079065607,\ x_{21} \to 0.6893762026321762,$
$x_{22} \to 0.006004759640005697,\ y_{11} \to 0.08149846432361406,\ y_{12} \to 0.0004975961200022679,$
$y_{21} \to 0.6760636867772296,\ y_{22} \to 0.005628781137626068,\ z_{11} \to 10.340984312404375,$
$z_{12} \to 14.427717115546843,\ z_{21} \to 66.06041961944968,\ z_{22} \to 49.74223524965058\}$
$\{23.9392\}$

The actual marginal cost is $1/M_3$.

$\{1/M_3\}/.\{F_3 \to 0.12086015700510068,\ G_3 \to 0.02113212603605529,\ M_3 \to 0.05712764322360773,$
$p_{31} \to 35.07799741549837,\ p_{32} \to 5678.541808898278,\ q_{32} \to 1.206568950723324^{-29},$
$U_3 \to 2.9631551816073056,\ V_3 \to 1.0537259131402723,\ w_{31} \to 0.00033879335219921813,$
$w_{32} \to 74.99408293177184,\ x_{31} \to 9.684934322017799^{-6},\ x_{32} \to 0.02113212587144456,$
$y_{31} \to 0.12085047207077866,\ y_{32} \to 1.646107289520379^{-10},\ z_{31} \to 119.9996612066478,$
$z_{32} \to 0.005917068228162191,\ F_1 \to 1.408211590367411,\ F_2 \to 1.365439889409406,$
$G_1 \to 0.0020946023279088287,\ G_2 \to 0.011633540777631766,\ M_1 \to 2.0437503246274403^{-19},$
$M_2 \to 0.05425525266887062,\ p_{11} \to 37.175585332603234,\ p_{12} \to 6686.675930253387,$
$p_{21} \to 54.16966546761761,\ p_{22} \to 8769.163943093063,\ q_{12} \to 0.2540209672621788,$
$q_{22} \to 0.7228466215440974,\ U_1 \to 1.3132899342122344,\ U_2 \to 1.439111016814532,$
$V_1 \to 1.4034789404061154,\ V_2 \to 13.839800717731972,\ w_{11} \to 49.65901568759563,$
$w_{12} \to 10.572282884453157,\ w_{21} \to 23.939580380550325,\ w_{22} \to 0.25776475034942337,$
$x_{11} \to 1.326713126043797,\ x_{12} \to 0.0015970062079065607,\ x_{21} \to 0.6893762026321762,$
$x_{22} \to 0.006004759640005697,\ y_{11} \to 0.08149846432361406,\ y_{12} \to 0.0004975961200022679,$
$y_{21} \to 0.6760636867772296,\ y_{22} \to 0.005628781137626068,\ z_{11} \to 10.340984312404375,$
$z_{12} \to 14.427717115546843,\ z_{21} \to 66.06041961944968,\ z_{22} \to 49.74223524965058\}$
$\{17.5047\}$

In this example the lower bound $B^2_{\min}$ is negative.

Finally, we use the simulated observations to estimate an upper bound on the coefficient of resource utilization, ρ, in the second period. The estimation procedure described in Appendix D defines an infinite dimensional optimization problem, i.e., the choice variables are utility functions of households and production functions of firms consistent with the market data. In fact, we can reduce the computation of $\rho_{\min}$ and $\rho_{\max}$ to polynomial programming problems where now the choice variables are simply vectors in some appropriate Euclidean space, $\mathbb{R}^N$. The "trick" is to use the second welfare theorem of general equilibrium theory. For the two-sector or more generally the Arrow–Debreu general equilibrium model, the theorem states that an economic state is Pareto optimal if and only if this state can be realized as a competitive equilibrium with lump-sum transfers of income between households.

Recall that Debreu's theorem on the coefficient of resource utilization for the two-sector model states that the economic state, where the social endowments of capital and labor are ρK and ρL, is Pareto optimal for the original utility levels. See Figure 4.

Consequently, we simply augment the equilibrium inequalities with the Walrasian inequalities, characterizing a competitive equilibrium, derived by Brown and Matzkin (1966), where now the competitive prices for goods and factors are also unknowns. In the Walrasian inequalities, social endowments are tK and tL, where t is unknown but constrained to be between 0 and 1. We also need the inequalities given in (16) and (17) defining the community indifference curve in Figure 4.

$$
\begin{aligned}
&\text{NMaximize}[\{0, U_1 > 1,\ U_2 > 1,\ U_3 > 1,\ x_{11} > 0,\ x_{12} > 0,\ x_{21} > 0,\ x_{22} > 0,\ x_{31} > 0,\ x_{32} > 0,\\
&V_1 > 1,\ V_2 > 1,\ V_3 > 1,\ y_{11} > 1/10000,\ y_{12} > 0,\ y_{21} > 0,\ y_{22} > 0,\ y_{31} > 0,\ y_{32} > 0,\ F_1 > 0,\\
&F_2 > 0,\ F_3 > 0,\ w_{11} > 0,\ w_{12} > 0,\ w_{21} > 0,\ w_{22} > 0,\ w_{31} > 0,\ w_{32} > 0,\ G_1 > 0,\ G_2 > 0,\\
&G_3 > 0,\ z_{11} > 0,\ z_{12} > 0,\ z_{21} > 0,\ z_{22} > 0,\ z_{31} > 0,\ z_{32} > 0,\ p_{11} > 0,\ p_{12} > 0,\ p_{21} > 0,\\
&p_{22} > 0,\ p_{31} > 0,\ p_{32} > 0,\ q_{12} > 1/10000,\ q_{22} > 0,\ q_{32} > 0,\ M_1 > 0,\ M_2 > 1/10000,\\
&M_3 > 1/10000,\ 1 > t > 0,\ U_2 * \{p_{11}, p_{12}\}.\{x_{11}, x_{12}\} \le U_1 * \{p_{11}, p_{12}\}.\{8x_{21}, x_{22}\},\\
&U_1 * \{p_{21}, p_{22}\}.\{x_{21}, x_{22}\} \le U_2 * \{p_{21}, p_{22}\}.\{x_{11}, x_{12}\},\\
&V_2 * \{p_{11}, p_{12}\}.\{y_{11}, y_{12}\} \le V_1 * \{p_{11}, p_{12}\}.\{y_{21}, y_{22}\},\\
&V_1 * \{p_{21}, p_{22}\}.\{y_{21}, y_{22}\} \le V_2 * \{p_{21}, p_{22}\}.\{y_{11}, y_{12}\}, \{p_{11}, p_{12}\}.\{x_{11}, x_{12}\} = \{1, q_{12}\}.\{60, 0\},\\
&\quad \{p_{21}, p_{22}\}.\{x_{21}, x_{22}\} = \{1, q_{22}\}.\{90, 0\}, \{p_{11}, p_{12}\}.\{y_{11}, y_{12}\}\\
&\quad = \{1, q_{12}\}.\{0, 25\} + p_{11} * F_1 - \{1, q_{12}\}.\{w_{11}, w_{12}\},\ p_{21} * F_2 - \{1, q_{22}\}.\{w_{21}, w_{22}\} \ge 0,\\
&\{p_{21}, p_{22}\}.\{y_{21}, y_{22}\} = \{1, q_{22}\}.\{0, 50\} + p_{21} * F_2 - \{1, q_{22}\}.\{w_{21}, w_{22}\},\\
&F_1 \le F_2 + M_2 * \{1, q_{22}\}.(\{w_{11}, w_{12}\} - \{w_{21}, w_{22}\}),\ F_2 \le F_1 + M_1 * \{1, q_{12}\}.(\{w_{21}, w_{22}\}\\
&\quad - \{w_{11}, w_{12}\}),\\
&M_1 * \{1, q_{12}\}.\{w_{11}, w_{12}\} < F_1,\ M_2 * \{1, q_{22}\}.\{w_{21}, w_{22}\} < F_2, p_{21} * F_2 - \{1, q_{22}\}.\{w_{21}, w_{22}\} \ge 0,\\
&p_{11} * F_1 - \{1, q_{12}\}.\{w_{11}, w_{12}\} \ge 0,\ \{1, q_{22}\}.\{z_{21}, z_{22}\} * G_1 \le \{1, q_{22}\}.\{z_{11}, z_{12}\} * G_2,\\
&\{1, q_{12}\}.\{z_{11}, z_{12}\} * G_2 \le \{1, q_{12}\}.\{z_{21}, z_{22}\} * G_1, w_{11} + z_{11} = 60, w_{12} + z_{12} = 25,\\
&w_{21} + z_{21} = 90, w_{22} + z_{22} = 50, x_{11} + y_{11} = F_1,\ x_{12} + y_{12} = G_1, x_{21} + y_{21} = F_2, x_{22} + y_{22} = G_2,\\
&F_3 \le F_1 + M_1 * \{1, q_{12}\}.(\{w_{31}, w_{32}\} - \{w_{11}, w_{12}\}),\ F_1 \le F_3 + M_3 * \{1, q_{32}\}.(\{w_{11}, w_{12}\}\\
&\quad - \{w_{31}, w_{32}\}),
\end{aligned}
$$

$F_3 \leq F_2 + M_2 * \{1, q_{22}\}.(\{w_{31}, w_{32}\} - \{w_{21}, w_{22}\}),\ F_2 \leq F_3 + M_3 * \{1, q_{32}\}.(\{w_{21}, w_{22}\}$
$- \{w_{31}, w_{32}\}),$
$\{1, q_{12}\}.\{z_{31}, z_{32}\}*G_1 \leq \{1, q_{12}\}.\{z_{11}, z_{12}\}*G_3,\ \{1, q_{32}\}.\{z_{11}, z_{12}\}*G_3 \leq \{1, q_{32}\}.\{z_{31}, z_{32}\}*G_1,$
$\{1, q_{22}\}.\{z_{21}, z_{22}\}*G_3 \leq \{1, q_{22}\}.\{z_{31}, z_{32}\}*G_2,\ \{1, q_{32}\}.\{z_{31}, z_{32}\}*G_2 \leq \{1, q_{32}\}.\{z_{21}, z_{22}\}*G_3,$
$U_1 * \{p_{31}, p_{32}\}.\{x_{31}, x_{32}\} \leq U_3 * \{p_{31}, p_{32}\}.\{x_{11}, x_{12}\},$
$U_3*\{p_{11}, p_{12}\}.\{x_{11}, x_{12}\} \leq U1*\{p_{11}, p_{12}\}.\{x_{31}, x_{32}\},\ U_2*\{p_{31}, p_{32}\}.\{x_{31}, x_{32}\} \leq U_3*\{p_{31}, p_{32}\}.\{x_{21}, x_{22}\},$

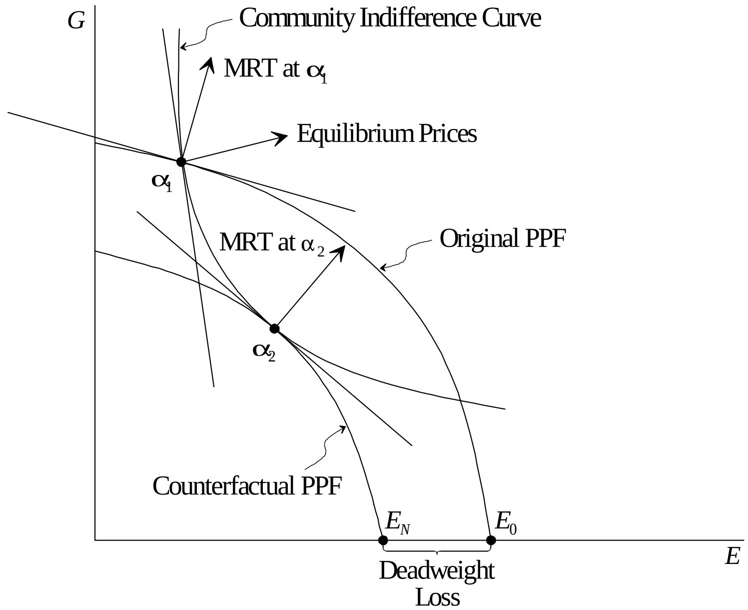
$U_3*\{p_{21}, p_{22}\}.\{x_{21}, x_{22}\} \leq U_2*\{p_{21}, p_{22}\}.\{x_{31}, x_{32}\},\ V_1*\{p_{31}, p_{32}\}.\{y_{31}, y_{32}\} \leq V_3*\{p_{31}, p_{32}\}.\{y_{11}, y_{12}\},$

$V_3*\{p_{11}, p_{12}\}.\{y_{11}, y_{12}\} \leq V_1*\{p_{11}, p_{12}\}.\{y_{31}, y_{32}\},\ V_2*\{p_{31}, p_{32}\}.\{y_{31}, y_{32}\} \leq V_3*\{p_{31}, p_{32}\}.\{y_{21}, y_{22}\},$

$V_3 * \{p_{21}, p_{22}\}.\{y_{21}, y_{22}\} \leq V_2 * \{p_{21}, p_{22}\}.\{y_{31}, y_{32}\},\ w_{31} + z_{31} = t * 90,\ w_{32} + z_{32} = t * 50,$
$\{p_{31}, p_{32}\}.\{x_{31}, x_{32}\} + \{p_{31}, p_{32}\}.\{y_{31}, y_{32}\} \leq t * (90 + 50q_{32}),\ x_{31} + y_{31} = F_3,$
$x_{32} + y_{32} = G_3,\ G_3 * p_{32} = \{1, q_{22}\}.\{z_{31}, z_{32}\},\ \ p_{31} * M_3 = 1.0,\ U_2 = U_3,\ V_2 = V_3\},$
$\{F_1,\ F_2,\ F_3,\ w_{11},\ w_{12},\ w_{21},\ w_{22},\ w_{31},\ w_{32},\ G_1,\ G_2,\ G_3,\ z_{11},\ z_{12},\ z_{21},\ z_{22},\ z_{31},$
$z_{32},\ U_1,\ U_2,\ U_3,\ x_{11},\ x_{12},\ x_{21},\ x_{22},\ x_{31},\ x_{32},\ V_1,\ V_2,\ V_3,\ y_{11},\ y_{12},\ y_{21},$
$y_{22},\ y_{31},\ y_{32},\ p_{11},\ p_{12},\ p_{21},\ p_{22},\ p_{31},\ p_{32},\ q_{12},\ q_{22},\ q_{32},\ M_1,\ M_2,\ M_3,\ t\}]$

$\{F_1 \to 1.408211590367411,\ F_2 \to 1.365439889409406,\ G_1 \to 0.0020946023279088287,$
$G_2 \to 0.011633540777631766,\ M_1 \to 2.0437503246274403^{-19},$
$M_2 \to 0.05425525266887062,\ p_{11} \to 37.175585332603234,\ p_{12} \to 6686.675930253387,$
$p_{21} \to 54.16966546761761,\ p_{22} \to 8769.163943093063,\ q_{12} \to 0.2540209672621788,$
$q_{22} \to 0.7228466215440974,\ U_1 \to 1.3132899342122344,\ U_2 \to 1.439111016814532,$
$V_1 \to 1.4034789404061154,\ V_2 \to 13.839800717731972,\ w_{11} \to 49.65901568759563,$
$w_{12} \to 10.572282884453157,\ w_{21} \to 23.939580380550325,\ w_{22} \to 0.25776475034942337$
$x_{11} \to 1.326713126043797,\ x_{12} \to 0.0015970062079065607,\ x_{21} \to 0.6893762026321762,$
$x_{22} \to 0.006004759640005697\ , y_{11} \to 0.08149846432361406,\ y_{12} \to 0.0004975961200022679,$
$y_{21} \to 0.6760636867772296,\ y_{22} \to 0.005628781137626068,\ z_{11} \to 10.340984312404375,$
$z_{12} \to 14.427717115546843,\ z_{21} \to 66.06041961944968,\ z_{22} \to 49.74223524965058\}$

A solution to this system of polynomial inequalities defines a family of utility functions, production functions, household demands, factor demands, prices for goods and factors and ρ, the value of t. The resulting two-sector model is in a cost minimizing market equilibrium in both periods. The counterfactual competitive equilibrium with lump-sum income transfers in the second period defines a Pareto optimal economic state for the social endowments ρK and ρL, where aggregate demands of E and G lie on the community indifference curve defined by values in the second period, see Figure 5. The system of inequalities with solution values are given next, where $\rho_{\min}$ and $\rho_{\max}$ are computed by doing a binary search on $[0, 1]$ with NMaximize.



$\rho = E_N / E_0$

$\alpha_1 \equiv$ aggregate equilibrium demands

$\alpha_2 \equiv$ aggregate demands in counterfactual competitive equilibrium with transfers

Figure 5

$$
\begin{aligned}
&\text{NMaximize}[\{0, U_3 > 1,\ x_{31} > 0,\ x_{32} > 0,\ V_3 > 1, y_{31} > 0,\ y_{32} > 0,\\
&\quad F_3 > 0,\ w_{31} > 0, w_{32} > 1/10000,\ G_3 > 0,\ z_{31} > 0,\ z_{32} > 1/10000,\ p_{31} > 0,\ p_{32} > 0,\\
&\quad M_3 > 1/10000,\ 0.95 \geq t,\ F_3 \leq 1.408211590367411 + 2.0437503246274403^{-19}\\
&\qquad (-49.65901568759563 + w_{31} + 0.2540209672621788(-10.572282884453157 + w_{32})),\\
&\quad 1.408211590367411 \leq F_3 + M_3(49.65901568759563 - w_{31} + q_{32}(10.572282884453157 - w_{32})),\\
&\quad F_3 \leq 1.365439889409406 + 0.05425525266887062(-23.939580380550325 + w_{31}\\
&\qquad + 0.7228466215440974(-0.25776475034942337 + w_{32})),\\
&\quad 1.365439889409406 \leq F_3 + M_3(23.939580380550325 - w_{31} + q32(0.25776475034942337 - w_{32})),\\
&\quad 0.0020946023279088287(z_{31} + 0.2540209672621788 z_{32}) \leq 14.005926969480676 G_3,\\
&\quad G_3(10.340984312404375 + 14.427717115546843 q_{32}) \leq 0.0020946023279088287(z_{31} + q_{32} z_{32}),\\
&\quad 102.0164263177113 G_3 \leq 0.011633540777631766(z_{31} + 0.7228466215440974 z_{32}),\\
&\quad 0.011633540777631766(z_{31} + q_{32} z_{32}) \leq G_3(66.06041961944968 + 49.74223524965058 q_{32}),\\
&\quad 1.3132899342122344(p_{31}x_{31} + p_{32}x_{32}) \leq (1.326713126043797 p_{31} + 0.0015970062079065607 p_{32}) U_3,\\
&\quad 60.U_3 \leq 1.3132899342122344(37.175585332603234 x31 + 6686.675930253387 x_{32}),\\
&\quad 1.439111016814532(p_{31}x_{31} + p_{32}x_{32}) \leq (0.6893762026321762 p_{31} + 0.006004759640005697 p_{32}) U_3,\\
&\quad 90.U_3 \leq 1.439111016814532(54.16966546761761 x_{31} + 8769.163943093063 x_{32}),\\
&\quad 1.4034789404061154(p_{31}y_{31} + p_{32}y_{32}) \leq (0.08149846432361406 p_{31} + 0.0004975961200022679 p_{32}) V_3,\\
&\quad 6.357017113545275 V_3 \leq 1.4034789404061154(37.175585332603234 y_{31} + 6686.675930253387 y_{32}),\\
&\quad 13.839800717731972(p_{31}y_{31} + p_{32}y_{32}) \leq (0.6760636867772296 p_{31} + 0.005628781137626068 p_{32}) V_3,\\
&\quad 85.98184834315961 V_3 \leq 13.839800717731972(54.16966546761761 y_{31} + 8769.163943093063 y_{32}),\\
&\quad w_{31} + z_{31} = t * 90,\ w_{32} + z_{32} = t * 50,\ p_{31}x_{31} + p_{32}x_{32} + p_{31}y_{31} + p_{32}y_{32} = t * (90 + 50 q_{32}),
\end{aligned}
$$

$x_{31} + y_{31} = F_3,\ x_{32} + y_{32} = G3,\ G_3 p_{32} = z_{31} + 0.7228466215440974 z_{32},$
$M_3 p_{31} = 1,\ 1.439111016814532 = U_3,\ 13.839800717731972 = V_3\},$
$\{t, F_3, w_{31}, w_{32}, G_3, z_{31}, z_{32}, U_3, x_{31}, x_{32}, V_3, y_{31}, y32, p_{31}, p_{32}, q_{32}, M_3\}]$

$\{0, \{t \to 0.982167,\ F_3 \to 1.41684,\ G_3 \to 0.0113304,\ M_3 \to 0.0206193,\ p_{31} \to 48.4983,$
$p_{32} \to 8723.42,\ q_{32} \to 1.61193,\ U_3 \to 1.43911,\ V_3 \to 13.8398, w_{31} \to 25.0529,$
$w_{32} \to -0.0000446665,\ x_{31} \to 0.720411,\ x_{32} \to 0.00582745,\ y_{31} \to 0.696433,$
$y_{32} \to 0.00550296,\ z_{31} \to 63.3422,\ z_{32} \to 49.1084\}\}$

$\{U_3 > 1,\ x_{31} > 0,\ x_{32} > 0,\ V_3 > 1,\ y_{31} > 0,\ y_{32} > 0,\ F_3 > 0,\ w_{31} > 0,$
$w_{32} > 1/10000,\ G_3 > 0,\ z_{31} > 0,\ z_{32} > /10000,\ p_{31} > 0,\ p_{32} > 0,\ M_3 > 1/10000,$
$0.95 \geq t,\ F_3 \leq 1.408211590367411 + 2.0437503246274403^{-19}$
$(-49.65901568759563 + w_{31} + 0.2540209672621788(-10.572282884453157 + w_{32})),$
$1.408211590367411 \leq F_3 + M_3(49.65901568759563 - w_{31} + q_{32}(10.572282884453157 - w_{32})),$
$F_3 \leq 1.365439889409406 + 0.05425525266887062$
$(-23.939580380550325 + w_{31} + 0.7228466215440974(-0.25776475034942337 + w_{32})),$
$1.365439889409406 \leq F_3 + M_3(23.939580380550325 - w_{31} + q_{32}(0.25776475034942337 - w_{32})),$
$0.0020946023279088287(z_{31} + 0.2540209672621788 z_{32}) \leq 14.005926969480676 G_3,$
$G_3(10.340984312404375 + 14.427717115546843 q_{32}) \leq 0.0020946023279088287(z_{31} + q_{32} z_{32}),$
$102.0164263177113 G_3 \leq 0.011633540777631766(z_{31} + 0.7228466215440974 z_{32}),$
$0.011633540777631766(z_{31} + q_{32} z_{32}) \leq G_3(66.06041961944968 + 49.74223524965058 q_{32}),$
$1.3132899342122344(p_{31} x_{31} + p_{32} x_{32}) \leq (1.326713126043797 p_{31} + 0.0015970062079065607 p_{32}) U_3,$
$60.U_3 \leq 1.3132899342122344(37.175585332603234 x_{31} + 6686.675930253387 x_{32}),$
$1.439111016814532(p_{31} x_{31} + p_{32} x_{32}) \leq (0.6893762026321762 p_{31} + 0.006004759640005697 p_{32}) U_3,$
$90.U_3 \leq 1.439111016814532(54.16966546761761 x_{31} + 8769.163943093063 x_{32}),$
$1.4034789404061154(p_{31} y_{31} + p_{32} y_{32}) \leq (0.08149846432361406 p_{31} + 0.0004975961200022679 p_{32}) V_3,$
$6.357017113545275 V_3 \leq 1.4034789404061154(37.175585332603234 y_{31} + 6686.675930253387 y_{32}),$
$13.839800717731972(p_{31} y_{31} + p_{32} y_{32}) \leq (0.6760636867772296 p_{31} + 0.005628781137626068 p_{32}) V_3,$
$85.98184834315961 V_3 \leq 13.839800717731972(54.16966546761761 y_{31} + 8769.163943093063 y_{32}),$
$w_{31} + z_{31} = t * 90, w_{32} + z_{32} = t * 50, p_{31} x_{31} + p_{32} x_{32} + p_{31} y_{31} + p_{32} y_{32} = t * (90 + 50 q_{32}),$
$x_{31} + y_{31} = F_3, x_{32} + y_{32} = G_3, G_3 p_{32} = z_{31} + 0.7228466215440974 z_{32},$
$M_3 p_{31} = 1, 1.439111016814532 = U_3, 13.839800717731972 = V_3\}/.$

$\{t \to 0.9821673551911791,\ F_3 \to 1.4168438722691818,\ G_3 \to 0.01133040688428589,$
$M_3 \to 0.0206192715347645,\ p_{31} \to 48.49831859064371,\ p_{32} \to 8723.415125275707,$
$q_{32} \to 1.6119294608757908,\ U_3 \to 1.439111016814532,\ V_3 \to 13.839800717731974,$
$w_{31} \to 25.052897012712904,\ w_{32} \to -0.00004466645402345648,$
$x_{31} \to 0.7204110709198741,\ x_{32} \to 0.005827451666845332,\ y_{31} \to 0.6964328013493079,$
$y_{32} \to 0.005502955217440558,\ z_{31} \to 63.34216495449321,\ z_{32} \to 49.10841242601298\}$

{True, True, True, True, True, True, True, True, False, True, True, True, True, True, True,
False, False, True, False, True, False, False, False, False, False, False, True, True,
True, False, True, False, True, True, False, True, True, False, True, True, True}

We now evaluate the family of inequalities.

{True, True, True, True, True, True, True, True, -0.00004466645402345648, 1/10000, True, True, True, True, True, True, False, 1.4168438722691818, 1.408211590367411, 1.408211590367411, 2.275594569911997, 1.4168438722691818, 1.4157323365908143, True, 0.15880590204227216, 0.15869285135580927, 0.38067274007132446, 0.29848389345084525, 1.1558876190604406, 1.1498593445690541, 1.6577965658013185, 1.6569744981176235, 112.64613823204785, 112.64598882755918, 86.34666100887192, 86.34616563046461, 123.43831648157204, 123.49817216725172, True, 114.77704479373246, 114.77728723293647, 87.97985001067835, 87.9795688810762, True, 1189.971646411582, 1189.9716464094197, 88.39506196720612, 88.39506196720612, True, 167.55438830041962, 167.55428673436208, True, True, 98.83984278990754, 98.84001496603086, True, True, True}. $\rho_{\max} \simeq 0.98$.

It is important to appreciate that these bounds grow sharper with n, the number of observations. That is, the equilibrium inequalities grow at the rate $O(n^2)$, hence diminishing the set of utility functions and production functions consistent with the observed market data.

The corresponding partial equilibrium analysis is illustrated in Figure 6.

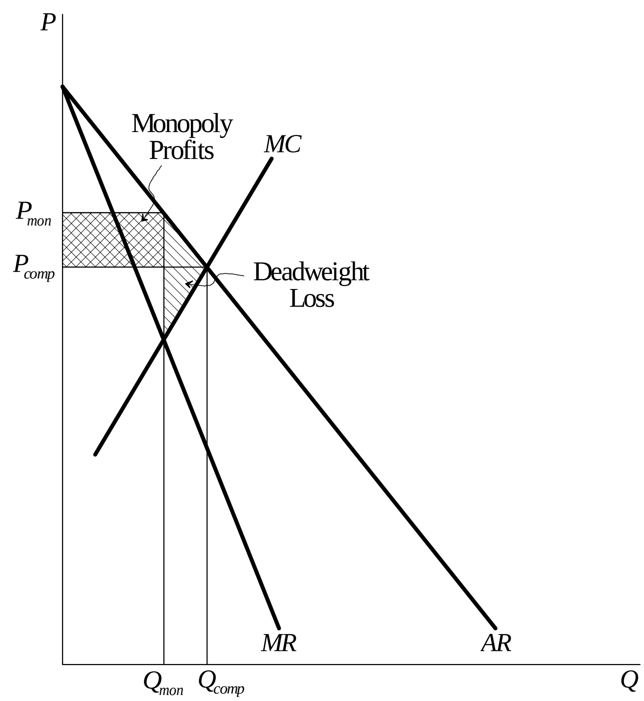


Figure 6

Acknowledgments

We are pleased to thank Bill Brainard, Al Klevorick and T.N. Srinivasan for their insights and comments. The generous support of the International Center for Finance and the able assistance of Glena Ames is also appreciated. The computations in

Appendix E were done on Grid Mathematica with much help from SOM's information technology (IT) group.

References

Afriat, S. (1967). "The Construction of Utility Function from Expenditure Data," *International Economic Review*, 66–77.

Bator, F. (1957). "The Simple Analytics of Welfare Maximization," *American Economic Review*, 22–59.

Brown, D., and G. Heal (1983). "Marginal vs. Average Cost Pricing in the Presence of a Public Monopoly," *American Economic Review* 189–193.

Brown, D., and R. Matzkin (1996). "Testable Restrictions on the Equilibrium Manifold," *Econometrica*, 64: 1249–1262.

Dierker, E., and B. Grodal (1998). "Modeling Policy Issues in a World of Imperfect Competition," *Scand. J. of Economics*, 153–179.

Fox, E., L. Sullivan and R. Peritz (2004). *Cases and Materials on Antitrust in Global Context.* Thomson and West Press.

Gavil, A., W. Kovacic and J. Baker (2002). *Antitrust Law in Perspective: Cases, Concepts and Problems in Competition Policy.* Thomson and West Press.

Gallhorn, E., and W. Kovacic (1994). *Antiturst Law and Economics in a Nutshell.* West Group Press.

Hylton, K. (2003). *Antiturst Law.* Cambridge University Press.

Morgan, D. (2001). *Cases and Materials on Modern Antitrust Law and its Origins.* West Group Press.

Posner, R. (2001). *Antitrust Law.* University of Chicago Press.

Varian, H. (1982). "The Nonparametric Approach to Demand Analysis," *Econometrica*, 50, 945–973.

_____ (1984). "The Nonparametric Approach to Production Analysis," *Econometrica*, 52, 579–597.

Wolfram, S. (2003). *The Mathematica Book 5.0.* Wolfram Media.