Is Noise Trading Cancelled Out by Aggregation?

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Abstract

Conventional wisdom suggests that investors' independent biases should cancel each other out and have little impact on equilibrium at the aggregate level. In contrast to this intuition, this paper analyzes models with biased investors and finds that biases often have a significant impact on the equilibrium even if they are independent across investors. First, independent biases affect the equilibrium asset price if investor demand for the asset is a nonlinear function of the bias. Second, even if the demand function is linear in the bias, it may still have a significant impact on the equilibrium due to the fluctuation of the wealth distribution. An initial run-up of the stock price makes optimistic investors richer, which then further pushes the stock price up and leads to lower future returns. This effect can lead to price overshooting, i.e., a negative expected future return. Similarly, an initial drop of the stock price leads to higher future returns. Simple calibrations show that a modest amount of biases can have a large impact on the equilibrium.

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1 Introduction

Although it is almost beyond debate that individual investors have biases, there is less consensus as to whether these biases have a significant impact on equilibrium at the aggregate level. One prevalent argument is that if biases are independent across investors, they generally should not have a large impact on equilibrium since they would cancel each other out. Although, to my knowledge, the reasoning behind this aggregation argument has never been formally spelled out in the literature, the argument is so intuitively appealing that it is not only often quoted casually in seminars and at conferences, but also frequently referred to in the literature (see Shleifer (2000), Hirshleifer (2001), Jeanee and Rose (2002), Ross (2004), Fehr and Tyran (2005), to name a few).

This aggregation argument has shaped our research discipline: *If* this argument holds, then to analyze equilibrium aggregate quantities and asset prices, economists can safely ignore individual biases, at least independent biases. This indeed is the premise of the mainstream asset-pricing literature. More interestingly, this argument is also the premise of the growing behavioral finance literature, which argues that individual biases tend to be correlated and so cannot be cancelled out by aggregation (see, e.g., Shleifer (2000), Hirshleifer (2001)).

Despite its importance, this aggregation argument has been taken for granted by *both* the traditional rational expectations paradigm and the recent behavioral literature. My paper fills this gap by directly examining this aggregation argument. My findings suggest that individual biases often have a significant impact on equilibrium even if they are independent across investors and the population average belief is unbiased. To understand the intuition behind these results, let me first recall the conventional aggregation argument. Suppose an unbiased investor's demand for a stock is D, which presumably is derived from utility maximization and depends on the price of the stock and some other parameters. There are N investors, and investor i's demand is $D_i = D + \epsilon_i$, where investor i's bias ϵ_i is a realization from $\tilde{\epsilon}$, a random variable with a mean of 0. If ϵ_1, ϵ_2,...,ϵ_N are independent realizations and N is large, then the average demand is approximately D, which is the average demand in the case without biases. As a result, these biases have little impact on the equilibrium stock price.

There are two cases where this argument fails. First, the argument implicitly assumes that the bias affects demand in a linear way, i.e., that D_i is a linear function of ϵ_i. The aggregation argument fails when the demand function is nonlinear in the bias. If, for example, each investor's demand is a convex function of his bias, then these biases increase the aggregate demand and so increase the stock price, even if the biases are independent across investors and the population average belief is unbiased. Similarly, biases decrease the stock price if each investor's demand function is concave in his bias. This is analogous to Jensen's inequality: if x is a random variable and f is a convex function, then $E[f(x)] > f[E(x)]$; if f is a concave function, then $E[f(x)] < f[E(x)]$. In order to elaborate further on the above intuition and to evaluate its implications quantitatively, this paper also analyzes three examples based on a typical demand function. The impact of independent biases is shown to be substantial and critically depends on the variable that investors are biased about: Suppose the demand function is linear in x. If investors' biases are about x, then the traditional aggregation argument would work since these biases enter the demand function in a linear way. If investors' biases are about $1/x$, however, the traditional aggregation argument would fail. This result makes some seemingly trivial questions extremely important. For example, do bond traders think in terms of prices or yields? This may seem trivial since there is a one-to-one relation between bond price and bond yield. However, in my model, the answer to this question can be very important: independent biases may have a large impact in one case but not the other.[1] The argument in the analysis in this case is

[1]In a recent study, Goldreich (2005) convincingly demonstrates that large dealers in U.S. Treasury auctions appear to be thinking in terms of yields rather than prices when they submit their bidding.

straightforward. However, the importance of this analysis is that it demonstrates that the impact of independent biases can be *quantitatively large*.

Second, even if biases affect demand in a linear way, they may still have a significant impact on the equilibrium as the wealth distribution fluctuates. The intuition is as follows. Suppose that there are two investors, A and B, and that both have the same initial wealth. A is optimistic about a stock and B pessimistic. At the initial date, relative to an unbiased investor, A holds more of the stock, and B less. If each investor's demand is a linear function of his bias, then one would expect that the biases do not affect the total demand from A and B and that the equilibrium stock price is not affected by these biases. This is essentially the traditional aggregation argument. However, this argument fails to hold in a dynamic setting. Suppose, after one period, the stock price goes up, say, due to some good news about the stock. Then the optimistic investor A has a larger wealth share relative to B, since A chose to hold more stock in the previous period. This fluctuation of the wealth distribution affects the equilibrium, and this effect is absent in models without biases.

The fluctuation of the wealth distribution causes stock return predictability. An initial run-up of the stock price makes optimistic investors richer, which then pushes the stock price further up and leads to lower future returns. Similarly, an initial drop of the stock price leads to higher future returns. As a result, these biases induce mean reversion in stock returns, and the fluctuations of the wealth distribution also amplify the shocks and lead to high volatility.[2]

It is interesting to compare this paper with the literature on short sale constraint and differences of opinion (e.g., Miller (1977), Harrison and Kreps (1978), Scheinkman and Xiong (2003)). This literature implies that short sale constraints generally lead

[2]The literature has documented evidence of short-run momentum and long-run reversal (e.g., De Bondt and Thaler (1985), Jegadeesh and Titman (1993)). The mean-reversion implication from my model might have contributed to the long-run reversal. This conjecture can be empirically tested in a panel data set of investors' disaggregate holdings.

to overvaluation but not undervaluation. This is in contrast to the implication in my model: idiosyncratic biases can amplify the fluctuations and induce mean reversion in stock returns. That is, idiosyncratic biases can lead to both overvaluation and undervaluation, depending on the past stock price movements.

Note also that, in my model, idiosyncratic biases can lead to stock price overshooting in the sense that the expected future return is negative. This is in sharp contrast to the literature on wealth fluctuation induced by heterogeneous preferences (e.g., Dumas (1989)) because in settings with rational expectations, preference heterogeneity alone cannot cause overshooting. It is also interesting to note that stock price overshooting arises without short sales constraints. A negative expected return is sustained in equilibrium: Optimistic investors are willing to take long positions in the stock because their perceived expected return is still positive. Pessimistic investors take short positions; however, due to risk aversion, their positions are limited. After a big increase in the stock price, the optimistic investors have most of the wealth in the economy, and so their influence dominates in equilibrium and leads to stock price overshooting. A simple calibration exercise shows that a modest amount of independent biases can have a large impact on the equilibrium and cause stock price overshooting.

The impact of wealth fluctuation has long been studied in the literature (e.g., Dumas (1989)). The contribution of the current paper is to show that this familiar mechanism in the literature actually provides a strong argument *against* the commonly held view on the impact of independent biases, an issue that is fundamental to economics and finance.

A number of empirical studies have shown that individual investors' trades are correlated, and documenting the impact of these trades on asset prices (see, e.g., Barber, Odean and Zhu (2006, 2009), Kaniel, Saar and Titman (2008), among others). These studies indicate that investors' biases are not independent, and the correlated biases might have a large impact on asset prices. My paper complements this literature by pointing out the flaws in the conventional aggregation argument and demonstrating that biases can have a significant impact on asset prices even if they are independent across investors. Moreover, the underlying argument can be taken further to analyze asset price dynamics. For example, Xiong and Yan (2009) analyze a model in which investors have biased beliefs but their average belief coincides with the rational belief. The wealth fluctuation induced by biases can explain a number of stylized facts about bond prices (e.g., the high volatility of long-term bond yields and the strong time variation in bond return premium). More recently, Jouini and Napp (2010) substantially generalize my continuous-time model in the appendix to study the impact on state price density.

The rest of the paper is organized as follows. Section 2 presents a one-period model to illustrate that the aggregation argument may fail if investors' demand functions are nonlinear in the bias. Section 3 presents a two-period model to illustrate that even if investors' demand functions are linear in the bias, the aggregation argument may still fail due to the wealth share fluctuation. Section 4 concludes. Appendix A reports the proofs, and Appendix B presents a continuous-time version of the model in Section 3 to illustrate robustness.

2 A Static Model

My goal is to set up models with biased investors to formalize the intuition outlined in the introduction. As pointed out in Barberis and Thaler (2003), there are two major approaches in modeling biases: the preference-based approach and the belief-based approach. In the preference-based approach, the investors are assumed to have nonstandard preferences to reflect their biased behavior. In the belief-based approach, however, biased behavior is captured by assuming that investors have biased beliefs. Despite the stark difference in modeling, these two approaches can lead to similar predictions (see, e.g., Basak and Yan (2007, 2009)). In this paper, I adopt the belief-based formulation of biases. One advantage of this approach is that it allows me to adopt the modeling tools developed in the heterogenous beliefs literature. This advantage is especially clear in the dynamic model in Section 3.

Consider a one-period (two dates) model with $t = 0, 1$. There are two assets in the economy: a riskless bond and a stock. Given that the focus of this paper is on the impact of noise trading on the stock market, the riskless bond is assumed to be in perfectly elastic supply and the interest rate r_f is set exogenously. The stock, which is normalized to one share, is a claim to a positive dividend v at $t = 1$. There are N investors, and each is endowed with $1/N$ share of the stock and no bond. At $t = 0$, investors make portfolio decisions and the stock price P_0 is determined in the equilibrium by equating the aggregate demand for the stock to the supply.

Investor i is assumed to allocate a fraction θ_i of his wealth to the stock market:

$$\theta_i = \theta\left(P_0, \Psi, \epsilon_i\right), \tag{1}$$

where Ψ includes parameters such as those for the investor's preference and the distribution of v_1, ϵ_i denotes investor i's bias, and $\epsilon_i = 0$ corresponds to the case in which investor i is unbiased. The unbiased investor's decision rule $\theta\left(P_0, \Psi, 0\right)$ can be interpreted as the one that maximizes his expected utility. $\theta\left(P_0, \Psi, \epsilon_i\right)$ can be interpreted as the optimal decision from the perspective of an investor with a bias ϵ_i. That is, all investors are identical, except that due to each one's idiosyncratic bias ϵ_i, their demand for the stock has an idiosyncratic component.

The demand function θ is assumed to satisfy the following conditions:

$$\frac{\partial \theta}{\partial P_0} < 0, \tag{2}$$

$$\lim_{P_0 \to \infty} \theta = -\infty, \tag{3}$$

$$\lim_{P_0 \to 0} \theta = \infty. \tag{4}$$

The above condition (2) implies that investors demand less stock if the stock price is higher. For simplicity, technical conditions (3) and (4) are made to ensure the existence and uniqueness of the equilibrium.

Investors' biases, ϵ_i for $i = 1, ..., N$, are independent realizations from $\tilde{\epsilon}$, which is a random variable with

$$E[\tilde{\epsilon}] = 0, \tag{5}$$

$$E[\theta(P_0, \Psi, \tilde{\epsilon})] < \infty. \tag{6}$$

Intuitively, this assumes that the biases are independent across investors, and equation (5) implies that the population on average is unbiased. Technical condition (6) is to make sure the average demand is well behaved. Finally, instead of exogenously specifying a demand function, one can specify a preference and derive the demand function endogenously by maximizing expected utility. This approach leads to similar results, while the analysis becomes more cumbersome.

Consider first the benchmark case in which all investors are unbiased, i.e., $\epsilon_i = 0$ for $i = 1, ..., N$. In this case, it is straightforward to see that the market clearing condition implies that the equilibrium stock price P_0^* satisfies

$$\theta(P_0^*, \Psi, 0) = 1. \tag{7}$$

Let's now consider the case with biased investors. It is natural to say that the conventional aggregation argument works and biases have no impact on the price if the equilibrium price in this economy with biased investors is still P_0^*. As formalized in the following proposition, however, this conventional aggregation argument holds only under a stringent condition.

Proposition 1 *In the economy described above, when the number of investors, N, goes to infinity, idiosyncratic biases have no impact on the stock price if and only if*

$$E[\theta(P_0^*, \Psi, \tilde{\epsilon})] = 1. \tag{8}$$

The above proposition shows that the conventional aggregation argument requires the strict condition (8), which, combined with (7), can be rewritten as

$$E[\theta\left(P_0^*, \Psi, \tilde{\epsilon}\right)] = \theta\left(P_0^*, \Psi, E[\tilde{\epsilon}]\right).$$

That is, the aggregation argument works if the bias $\tilde{\epsilon}$ enters the demand function $\theta(\cdot)$ in a special form so that $E[\theta(\tilde{\epsilon})] = \theta(E[\tilde{\epsilon}])$. This condition is quite stringent and generally does not hold. In the special case where $\theta(\cdot)$ is linear in $\tilde{\epsilon}$, condition (8) holds and the traditional aggregation argument holds. If $\theta(\cdot)$ is convex in $\tilde{\epsilon}$, however, $E[\theta\left(P_0^*, \Psi, \tilde{\epsilon}\right)] > \theta\left(P_0^*, \Psi, 0\right)$. That is, the biases increase the overall demand, and so increase the stock price. Similarly, if $\theta(\cdot)$ is concave in $\tilde{\epsilon}$, the biases decrease the overall demand and the stock price. Finally, note that although $\theta(\cdot)$ being linear in $\tilde{\epsilon}$ is sufficient for the aggregation argument to hold, it is not a necessary condition. One can imagine that if $\theta(\cdot)$ is concave in $\tilde{\epsilon}$ in some domains, but convex in some other domains, it is possible to have the knife-edge case where condition (8) holds.

2.1 Examples

In order to quantitatively illustrate the impact from independent biases, let's now consider several examples by specifying the demand function (1). The dividend from the stock, v, is now assumed to be lognormally distributed: $\ln v \sim \mathcal{N}(\bar{v}, \sigma^2)$. Denote the realized stock return as $r \equiv \ln \frac{v}{P_0}$.

An unbiased investor's demand function for the stock is assumed to be

$$\theta = \frac{E\left[r\right] - r_f}{\sigma^2} + \frac{1}{2}, \tag{9}$$

where $E\left[r\right]$ denotes the expected stock return. The decision rule (9) is approximately optimal for an investor with logarithmic preference (see Campbell and Viceira (1999)).

Let's first consider the case without bias. If all investors follow decision rule (9), one can easily obtain from the market clearing condition that the stock price at $t = 0$

is given by

$$P_0^* = \exp\left(\bar{v} - r_f - \frac{1}{2}\sigma^2\right), \tag{10}$$

and the expected stock return is

$$E[r^*] = r_f + \frac{1}{2}\sigma^2. \tag{11}$$

To illustrate the impact of individual biases, this paper analyzes three examples in which investors are biased about the parameters of the dividend: $\bar{v}$ and σ.

2.1.A Example 1

In this example, investors are assumed to be biased about $\bar{v}$. Hence, investors are biased about the expected stock return:

$$E^i[r] = E[r] + \phi\epsilon_i,$$

that is, investor i thinks the expected return is $E[r] + \phi\epsilon_i$, where $\phi \geq 0$, and ϵ_i, for $i = 1, ..., N$, are independent realizations from $\tilde{\epsilon}$, which is uniformly distributed on $[-1, 1]$. That is, investors have independent biases and the population average belief is unbiased. The most optimistic investor overestimates the expected return by ϕ while the most pessimistic investor underestimates the expected return by ϕ. Due to his bias, investor i allocates a fraction θ_i of his wealth to the stock market:

$$\theta_i = \frac{E^i[r] - r_f}{\sigma^2} + \frac{1}{2}. \tag{12}$$

Note that each investor's demand function (12) is linear in the bias. Hence, when N is large, individual biases are cancelled out by aggregation, and the stock price is still given by (10). Although the biases have no impact on equilibrium at the aggregate level, they may affect each investor's wealth: investor i's wealth at $t = 1$ is given by

$$W_i = \frac{v}{N} + \frac{\phi P_0^*}{N\sigma^2}\left(e^r - e^{r_f}\right)\epsilon_i. \tag{13}$$

If an investor is unbiased (i.e., $\epsilon_i = 0$) his wealth at $t = 1$ is v/N. A biased investor's wealth is generally different unless the stock return happens to be the same as the bond return (i.e., $r = r_f$). In particular, if the stock outperforms the bond, the optimistic investors are richer relative to pessimistic investors. Similarly, the pessimistic investors become relatively richer if the bond outperforms the stock.

2.1.B Example 2

In this example, investors are assumed to be biased about σ: investor i thinks the standard deviation of the stock return is

$$\sigma_i = \sigma + \phi\epsilon_i,$$

where $0 \leq \phi < \sigma$, ϵ_i for $i = 1, ..., N$, are independent realizations from $\tilde{\epsilon}$, which is uniformly distributed on $[-1, 1]$. The biases are independent across investors, and the most optimistic investor underestimates the standard deviation by ϕ, while the most pessimistic investor overestimates the standard deviation by ϕ. As a result, investor i allocates a fraction θ_i of his wealth to the stock market:

$$\theta_i = \frac{E[r] - r_f}{\sigma_i^2} + \frac{1}{2}. \tag{14}$$

The conventional aggregation argument fails here because the demand (14) is convex in the bias. These biases increase the aggregate demand and so increase the equilibrium stock price and decrease the expected future return: if investors follow the decision rule (14), the expected stock return is given by

$$E[r] = r_f + \frac{1}{2}\sigma^2 - \frac{1}{2}\phi^2. \tag{15}$$

Compared with equation (11), the above expression shows that the biases reduce the expected stock return by $\frac{1}{2}\phi^2$. Suppose $\sigma = 0.25$ and $\phi = 0.1$, that is, the true volatility is 25% and investors' beliefs range from 15% to 35%. Then these biases reduce the expected stock return by 0.5%.

Merton (1980) points out that with high-frequency data one can estimate volatility accurately. This argument does not nullify Example 2, which assumes investors are biased about the standard deviation of the stock return. Rather, it helps to identify situations where this argument is relatively more important. For example, in new industries where there is not much data, one might conjecture that people will have different opinions about volatility. Moreover, people may have different opinions about the volatility of an industry if the uncertainty of the industry tends to change dramatically. Therefore, Example 2 might be more relevant for these industries. Moreover, even when the bias for volatility is small, its impact may still be amplified by the interaction between the bias for expected return and the bias for volatility, as illustrated in the next example.

2.1.C Example 3

Suppose investors are biased about both the expected return and the standard deviation of the stock:

$$E^i[r] = E[r] + \phi_1 \left(\sqrt{1-\rho^2}\epsilon_{1i} + \rho\epsilon_{2i} \right), \tag{16}$$

$$\sigma_i = \sigma + \phi_2 \epsilon_{2i}, \tag{17}$$

where $0 \leq \phi_1$, $0 \leq \phi_2 < \sigma$. For $i = 1, ..., N$, ϵ_{1i} are independent realizations from $\tilde{\epsilon}_1$, and ϵ_{2i} are independent realizations from $\tilde{\epsilon}_2$. $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ are independent and uniformly distributed on $[-1, 1]$. ρ captures the correlation between an investor's bias on the expected return and his own bias on the volatility. That is, investor i thinks the expected stock return is $E^i[r]$ and the standard deviation is σ_i. These biases are independent *across investors*. But for each investor, his two biases might be correlated. In the case of $\rho > 0$, for example, if an investor overestimates the volatility, then he also tends to overestimate the expected return. The biases in (16)–(17) imply that investor i allocates a fraction θ_i of his wealth to the stock market:

$$\theta_i = \frac{E^i[r_1] - r_f}{\sigma_i^2} + \frac{1}{2}. \tag{18}$$

If investors follow the decision rule (18), the expected stock return is given by

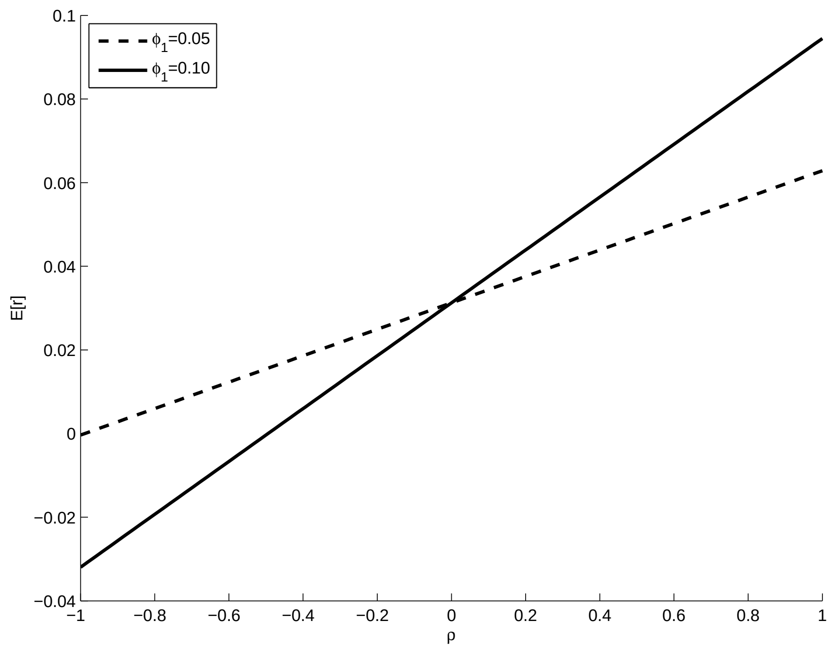
$$E[r] = r_f + \frac{1}{2}\left(\sigma^2 - \phi_2^2\right) + \rho\phi_1 \left(\frac{\sigma}{\phi_2} + \frac{\sigma^2 - \phi_2^2}{2\phi_2^2} \log \frac{\sigma - \phi_2}{\sigma + \phi_2}\right). \tag{19}$$

The above result includes Examples 1–2 as two special cases: one can obtain the result in Example 1 by letting ϕ_2 go to 0 and obtain the result in Example 2 by setting $\phi_1 = 0$. If investors' biases on expected return are independent from their biases on volatility (i.e., $\rho = 0$) only the biases on volatility affect the stock price and the equilibrium is the same as that in Example 2.

More interesting results arise when the biases on expected return and the biases on volatility are correlated. Suppose $\sigma = 0.25$, $\phi_2 = 0.2$. That is, investors' beliefs about the volatility range from 5% to 45% when the true volatility is 25%. Figure 1 plots the expected stock return against the correlation ρ. It shows that the expected stock return increases substantially with respect to ρ. Suppose $\phi_1 = 5\%$, that is, investors' biases about the expected return range from an underestimation of 5% to an overestimation of 5%. The expected stock return increases from around 0% to 6.3% when ρ increases from -1 to 1. The impact of the correlation ρ is more significant when the biases about the expected return are larger. For example, in the case of $\phi_1 = 10\%$, when ρ increases from -1 to 1, the expected stock return increases from -3.2% to 9.3%. It is interesting to note that the expected stock return is negative when $\rho < -0.5$, even though there is no short sales constraint.

The underlying driving force here is similar to that in Example 2. If $\rho < 0$, θ_i is a convex function of ϵ_{2i} and the convexity increases the stock price and decreases the expected stock return. When this impact is strong enough, it can lead to a negative expected return. The economic intuition is also straightforward. In the case of $\rho <$

Figure 1: The expected stock return and the correlation. This figure plots the expected stock return, $E[r]$, against ρ, the correlation between the biases on the expected stock return and the biases on the volatility. For example, $\rho > 0$ implies that investors who overestimate the expected stock return also tend to overestimate the stock return volatility. Parameter values: $r_f = 2\%$, $\sigma = 0.25$, $\phi_2 = 0.2$.



0, for example, if an investor overestimates the expected stock return, he tends to underestimate the volatility. Hence this investor has a high demand for the stock. In contrast, if an investor underestimates the expected return, he may want to short the stock. However, his short position is limited since he also tends to overestimate the volatility and so feels the short position is risky. Therefore, the biases increase the aggregate demand and lead to a lower, or even negative, expected stock return. The negative expected stock return arises here in the equilibrium without short sales constraints. The pessimistic investors choose to limit their short position because they feel it is risky, not because it is prohibited.

Note also that the convexity decreases with respect to ρ, and hence the expected stock return increases with respect to ρ. Finally, θ_i may become concave in ϵ_{2i} when ρ is high enough, which explains why the biases can even increase the expected stock return when ρ is high enough: The expected stock return is 5.1% when all investors are unbiased. In the case of $\phi_1 = 10\%$, for example, the expected stock return is higher than 5.1% if $\rho > 0.32$.

2.2 Discussions of the static model

Examples 1–3 demonstrate that biases can have a substantial impact on equilibria even if they are independent across investors and the population average belief is unbiased. Moreover, the impact critically depends on the variable that investors are biased about. The argument here is straightforward, but the importance of these examples is that they show the impact of independent biases can be *quantitatively large*. These results suggest that the traditional aggregation argument might have understated the importance of individual biases for asset pricing.

Examples 1–3 have implications for cross-sectional stock returns. If investors are biased only about volatility, a stock's expected return tends to be lower if the biases are stronger. Moreover, if investors are biased about both the expected return and the volatility, the correlation between these two biases plays a key role in determining the expected return. Holding everything else constant, the higher the correlation between these two biases, the higher the expected stock return.

The main point of this section is to show that independent biases can have a large impact on the equilibrium if they enter demand functions in a nonlinear way. Therefore, the traditional aggregation argument should not be taken for granted. It is an important empirical question to examine which variable investors' decisions are based on and the correlation of across biases of a given investor, since as made clear in these examples, the impact of independent biases critically depends on the variable that investors are biased about: Suppose the demand function is linear in x. If investors' biases are about x, then the traditional aggregation argument would work. If investors' biases are about $1/x$, however, the traditional aggregation argument would fail. This result makes some seemingly trivial questions extremely important. For example, do bond traders think in terms of prices or yields? This question may seem trivial since there is a one-to-one relation between bond price and bond yield. However, price and yield may enter the demand function in different forms. As shown in my analysis, this can be very important: independent biases may have a large impact in one case but not the other. In a recent study, Goldreich (2005) convincingly demonstrates that large dealers in U.S. Treasury auctions appear to be thinking in terms of yields rather than prices when they submit their bidding. In the context of my model, empirical evidence like this can have a significant impact on prices.

In the above examples, the traditional aggregation argument works only if the biases enter investors' demand function in a linear way. The next section shows, however, that even these "linear" biases can significantly affect the stock price in a dynamic setting. As will become clear, this can be viewed as another example in which (8) is violated: through their impact on investors' wealth level, biases effectively enter the demand function in a nonlinear form and significantly affect the equilibrium prices.

3 A Dynamic Model

To make the intuition transparent, this section provides a two-period model. A continuous-time version of this model, while making the analysis rigorous, delivers similar results and is reported in Appendix B.

Consider an endowment economy with two periods (three dates) $t = 0, 1, 2$. The exogenous aggregate consumption supply at time t is given by C_t, with $C_0 > 0$, and for $t = 1, 2$

$$\ln C_t = \ln C_{t-1} + g_t^c,$$

where g_1^c and g_2^c are i.i.d. $N(\mu_c, \sigma_c^2)$. Although the literature traditionally treats the stock dividend process the same as the aggregate consumption process, there is a clear motivation to model them separately: in the U.S., for example, the correlation between quarterly real consumption growth and real dividend growth is only 0.05 (see Campbell (2003)). Thus, the stock is modelled as follows: there is a dividend stream D_t, $t = 0, 1, 2$. For $t = 1, 2$

$$\ln D_t = \ln D_{t-1} + g_t^D,$$

where g_1^D and g_2^D are i.i.d. $N(\mu_D, \sigma_D^2)$ and $D_0 = 1$. The stock, normalized to one share, is a claim to two dividends D_1 and D_2. One interpretation is that the stock is in positive supply: the dividend D_t is a fraction of the aggregate consumption C_t, and the rest of the economy produces $C_t - D_t$ each period. An alternative interpretation is that the stock is in zero net supply. The analysis remains the same for both interpretations. For simplicity, the shocks to dividends, g_t^D, and the shocks to consumption, g_t^c, are assumed to be independent. It is straightforward to introduce correlations, and this will not affect the discussions below.

There are $2N+1$ investors who are numbered as $-N, -N+1, ..., -2, -1, 0, 1, 2, ...N$. These investors have independent biases about the dividend growth rate g_t^D: For $i = -N, ..., N$, investor i thinks the growth rate is μ_D^i

$$\mu_D^i = \mu_D + \frac{i}{N}\phi. \tag{20}$$

That is, investors' beliefs spread evenly across $[\mu_D - \phi, \mu_D + \phi]$, and investor 0 has the correct belief. This can be interpreted as investor i thinking the growth rate is $\mu_D^i = \mu_D + \epsilon_i$, where ϵ_i is an independent realization of a uniformly distributed random variable on $[-\phi, \phi]$. When N is large, investors' biases spread evenly across $[-\phi, \phi]$. So, specification (20) is chosen for expositional simplicity. Moreover, the assumption that investors are not biased about the aggregate consumption simplifies analysis and is not essential.

Assume there exists a complete set of Arrow-Debreu securities. At time 0, wealth is evenly distributed among investors: each investor is endowed with $1/(2N+1)$ share of the aggregate wealth, i.e., the claim to the aggregate consumption. For $i = -N, ..., N$, investor i has the following objective function

$$\max E^i \left[\sum_{t=0}^{2} \beta^t \ln c_t^i \right],$$

where c_t^i is his consumption at t, β is the patience parameter $0 < \beta \leq 1$, and $E^i[\cdot]$ means the expectation is taken from investor i's point of view.

The definition of a competitive equilibrium is standard: In the equilibrium, each investor maximizes his objective function and the good and financial markets clear. Denote the equilibrium stock price at t ($t = 0, 1$) as P_t; then the stock return for the first period is

$$r_1 \equiv \ln \frac{D_1 + P_1}{P_0},$$

and the stock return for the second period is

$$r_2 \equiv \ln \frac{D_2}{P_1}.$$

3.1 The case with one investor

Let's first analyze the case where the economy is populated by investor i only. In this case, investor i consumes the aggregate consumption. The standard consumption-based asset-pricing formula implies that the gross riskless interest rates are given by

$$R_t = \beta^{-1} \exp\left(\mu_c - \frac{1}{2}\sigma_c^2\right), \text{ for } t = 0 \text{ and } t = 1, \tag{21}$$

and the stock prices are given by

$$P_0^i = D_0 \left(k_i + k_i^2\right), \tag{22}$$

$$P_1^i = D_1 k_i, \tag{23}$$

where

$$k_i = \beta \exp\left(\mu_D^i - \mu_c + \frac{\sigma_D^2 + \sigma_c^2}{2}\right). \tag{24}$$

The riskless interest rate increases in μ_c and decreases in σ_c^2 (equation (21)). Intuitively, a higher consumption growth rate μ_c implies lower marginal utility in the future and so makes bonds, which are claims to future consumption, less valuable. Hence, the interest rate increases in μ_c. Moreover, higher σ_c^2 increases the precautionary saving motive and so decreases the interest rates. Note also that k_i increases in μ_D^i and decreases in μ_c (equation (24)). Therefore, the stock price increases in the expected dividend growth rate but decreases in the aggregate consumption growth rate. This is intuitive: the stock price should be higher when higher future dividends are expected. Moreover, a higher consumption growth rate leads to higher discount rates for future consumption, and so decreases the stock price.

3.2 The case with many biased investors

Let's now move on to analyze the impact of the biases specified in (20). The equilibrium is solved in the appendix, and the following proposition reports the stock prices and the riskless interest rates.

Proposition 2 *In the economy with biases, the equilibrium stock price is given by*

$$P_t = \sum_{i=-N}^{N} \omega_t^i P_t^i, \text{ for } t = 0 \text{ and } t = 1, \tag{25}$$

where P_t^i, given by (22) and (23), is the stock price that would prevail in an economy with investor i only, and ω_t^i is investor i's wealth share and is given by

$$\omega_0^i = \frac{1}{2N+1}, \tag{26}$$

$$\omega_1^i = \frac{\lambda_i}{\sum_{j=-N}^{N} \lambda_j}, \tag{27}$$

$$\ln \lambda_i = \frac{\mu_D^i - \mu_D}{\sigma_D^2}\left(\ln \frac{D_1}{D_0} - \mu_D\right) - \frac{(\mu_D^i - \mu_D)^2}{2\sigma_D^2}. \tag{28}$$

The interest rates are given by (21).

Proof. See Appendix A. ■

This proposition illustrates that biases induce fluctuations in the wealth distribution and so affect the equilibrium. In particular, the stock price is the wealth share-weighted average of the stock prices that would prevail in economies in each of which there is only one investor (equation (25)). A similar result was obtained in the literature (e.g., Detemple and Murthy (1994), Xiong and Yan (2009)). However, these studies restrict asset returns to diffusion processes. That is, the asset returns are restricted to be conditionally normal. One contribution of my model is to show that the wealth-weighted average structure also holds for general distributions of asset returns.[3]

By assumption, investors have the same wealth share at $t = 0$ (equation (26)). The wealth shares at $t = 1$, however, are generally determined by the performance of the stock market. As illustrated in (27) and (28), an investor's wealth share tends to increase if he happens to be right *ex post*. For example, if an investor is optimistic ($\mu_D^i > \mu_D$) and the stock's performance happens to be better than average ($\ln \frac{D_1}{D_0} > \mu_D$), the first term of the right-hand side of equation (28), $\frac{\mu_D^i - \mu_D}{\sigma_D^2} \left(\ln \frac{D_1}{D_0} - \mu_D \right)$, implies that his wealth share tends to increase since he chose to hold more stock. Similarly, a pessimistic investor's wealth share tends to rise when the stock's performance happens to be worse than average.[4]

The wealth fluctuation leads to stock return predictability. Intuitively, after a high stock return in the first period, optimistic investors have larger wealth shares at $t = 1$. Hence, the wealth share-weighted average belief becomes optimistic, which further pushes up the stock price and leads to lower future returns. Similarly, a low

[3]Note that from equation (25), the stock return between $t = 0$ and $t = 1$ is not normally distributed.

[4]The last term in equation (28), $-\frac{(\mu_D^i - \mu_D)^2}{\sigma_D^2}$, reveals that biased investors' wealth shares tend to decrease on average because these investors' decisions are inferior to those of rational investors. Even though the investor with the correct belief would dominate the economy in the long run, the literature has shown that this process is extremely slow (see Yan (2008), Dumas, Kurshev and Uppal (2008)).

return in the first period implies that pessimistic investors have larger wealth shares at $t = 1$. Hence, the wealth share-weighted average belief becomes pessimistic, which further presses down the stock price and leads to higher future returns. This intuition is formalized in the proposition: The stock price at $t = 1$ is the wealth share-weighted average of each investor's valuation P_1^i (equation (25)). Note that P_1^i increases in μ_D^i, that is, optimistic investors' valuations are higher (equations (23)– (24)). A positive shock to the stock market increases the weights of optimistic investors who have higher valuations, and a negative shock increases the weights of pessimistic investors who have lower valuations. Hence, the wealth fluctuation further amplifies shocks to the stock market and increases stock return volatility.

Biases also induce mean reversion in stock returns. It is easy to see that when there is no bias, the expected stock return for the second period is a constant:

$$E_1 [r_2] = -\ln \beta + \mu_c - \frac{\sigma_c^2 + \sigma_D^2}{2}. \tag{29}$$

That is, the realized stock return in the first period has no impact on the expected future return. When there are biases, however, Proposition 1 implies that, from an unbiased investor's point of view, the expected stock return in the second period is

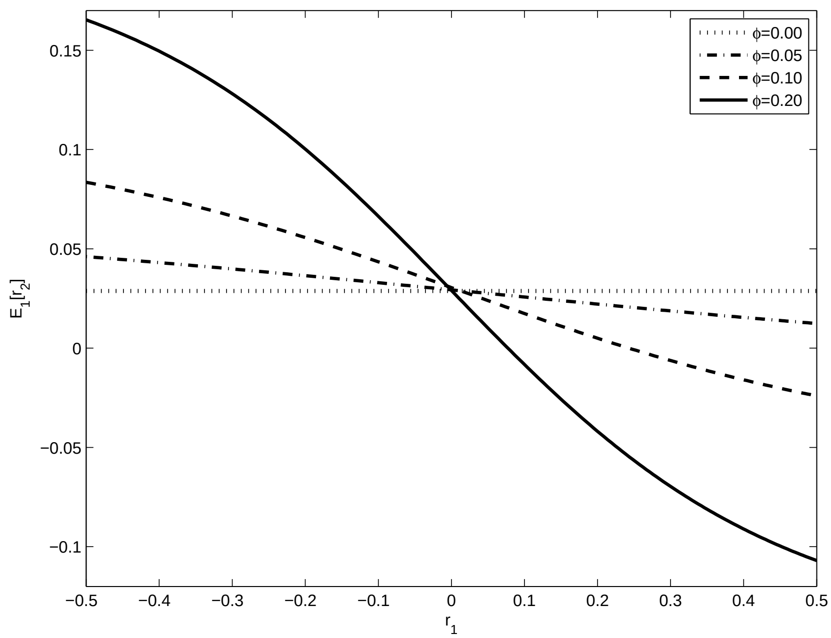
$$E_1 [r_2] = \ln \frac{1}{\sum_{i=-N}^{N} \omega_1^i k_i} + \mu_D. \tag{30}$$

It is worth emphasizing that the expected return $E_1 [r_2]$ refers to the expectation from an unbiased investor's point of view. That is, this is the average return that an econometrician would obtain if he had a long series of data.

Equation (30) reveals that the realization of the first period return affects the wealth distribution ω_1^i, and so affects the expected stock return for the second period. The magnitude of this effect is illustrated in the following calibration exercise. The model parameters are chosen as follows: $\mu_D = 2\%$, $\sigma_D = 15\%$, $\mu_c = 2\%$, $\sigma_c = 2\%$. These parameters are close to the estimates from the U.S. data (see, e.g., Campbell (2003)). Finally, the patience parameter $\beta = 0.98$ (i.e., one discounts next year's utility by

2%) and $N = 100$ (i.e., there are 201 investors in the economy). Varying these two parameters has little impact on the results below.

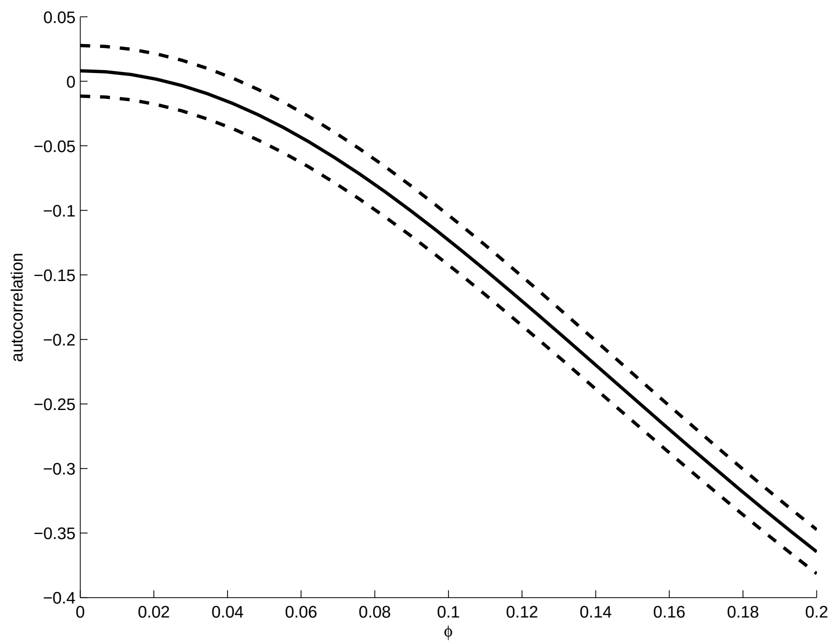
Figure 2: The expected future return and the past return. This figure plots the expected stock return in the second period, $E_1[r_2]$, against the realized stock return in the first period, r_1, for various values for ϕ. Parameter values: $\mu_D = 2\%$, $\sigma_D = 15\%$, $\mu_c = 2\%$, $\sigma_c = 2\%$, $\beta = 0.98$, and $N = 100$.



The stock return predictability is illustrated in Figure 2, which plots the expected stock return in the second period, $E_1[r_2]$, against the realized stock return in the first period r_1. In the presence of noise trading (i.e., $\phi > 0$), the realized return in the first period can predict the second period return. The plots are downward sloping, suggesting that a higher realized stock return in the first period implies a lower expected stock return in the second period. In the case of $\phi = 0.1$, for example, the plot shows that when the stock return in the first period increases from 0 to 20%, the expected stock return in the second period decreases from 2.9% to 0.5%. It is interesting to note that this mechanism can lead to stock price overshooting: if the stock return in the first period is higher than 24%, then the expected stock return in the second period becomes negative. Note that the negative expected return arises in equilibrium without short sales constraints. This is because the large run-up in the stock price makes the optimistic investors so rich, which further pushes up the stock price so much that the expected future return becomes negative. The comparison of the four cases in the figure shows that when the degree of biases increases (i.e., ϕ increases), the predictability becomes stronger (i.e., the plot becomes more downward sloping). In the case of $\phi = 0.2$, for example, when the stock return in the first period increases from 0 to 20%, the expected stock return for the second period decreases from 2.7% to -4.3%. Moreover, as suggested by equation (29), the plot for the case of $\phi = 0$ is flat, implying that the realized stock return for the first period does not predict the return for the second period. Finally, to get a sense of the degree of these biases, note that even in the case of $\phi = 0.2$, the most biased investors' biases are within two standard deviations of the dividend growth rate.

These biases also lead to a negative autocorrelation in stock returns. This is illustrated in Figure 3, which plots the autocorrelation of stock returns against the degree of bias ϕ. The economy is simulated 10,000 times, and the estimates of the autocorrelation coefficient are obtained by computing the correlation coefficient between the returns in the two periods across the 10,000 paths. The solid line plots the estimated autocorrelation coefficients, and the dotted lines are the lower and upper bounds for a 95% confidence interval for the estimates. The plot shows that these biases induce a negative correlation, and that the magnitude of the autocorrelation increases with the degree of bias. When there is no bias (i.e., $\phi = 0$), the autocorrelation is indifferent from 0. In the case of $\phi = 0.1$, however, the autocorrelation of stock returns is -0.13, highly significantly different from 0. The autocorrelation becomes as low as -0.38 when $\phi = 0.2$.

Figure 3: The autocorrelation in stock returns and biases. This figure plots the autocorrelation of stock returns against the degree of bias, ϕ. The economy is simulated 10,000 times, and the estimates of the autocorrelation coefficient are obtained by computing the correlation coefficient between the returns in the two periods across the 10,000 paths. The solid line plots the estimated autocorrelation coefficients, and the dashed lines are the lower and upper bounds for a 95% confidence interval for the estimates. Parameter values: $\mu_D = 2\%$, $\sigma_D = 15\%$, $\mu_c = 2\%$, $\sigma_c = 2\%$, $\beta = 0.98$, and $N = 100$.



Biases naturally lead to trading among investors, which can shed some light on the joint behavior of the asset prices and trading volume. For example, stronger biases (i.e., higher ϕ) make investors trade more. Hence, a higher ϕ leads to higher trading volume, higher volatility, and a stronger negative autocorrelation. This implication potentially provides a way to test the model. Given the difficulty in measuring ϕ directly, trading volume is a particularly useful instrument in measuring ϕ indirectly.

Finally, this proposition also shows that the biases have no impact on the riskless interest rate. This is due to the assumption that investors are not biased about the aggregate consumption growth rate. If biases about the aggregate consumption growth were introduced, they would induce interesting dynamics of the interest rates. See Xiong and Yan (2009) for a recent analysis of the impact of disagreement on the term structure of interest rates.

3.3 Discussion of the dynamic model

One notable feature of the model is that investors' beliefs do not change over time. In other words, it assumes that an investor's bias at $t = 0$ is the same as his bias at $t = 1$. This assumption is made for simplicity and can be relaxed. The implications discussed above still hold as long as investors' biases are persistent over time, that is, if one investor is optimistic at $t = 0$, he tends to be optimistic at $t = 1$ as well. The intuition is still the same: if some investors are optimistic at $t = 0$, then the run-up in stock price makes them richer at $t = 1$. Since these investors tend to be optimistic at $t = 1$, this further pushes up the stock price. Similarly, the aggregation argument also fails if investors' biases are negatively correlated over time: if one investor is optimistic at $t = 0$, he tends to be pessimistic at $t = 1$. Which case is more empirically plausible? Although the aggregation argument fails in both cases, the exact empirical implications are quite different. A detailed panel data set of investors' trading records might be able to offer an opportunity to measure the biases and shed some light on this question.

My dynamic model is related to the literature on the impact of the fluctuation of wealth distribution, induced by the heterogeneity in risk aversion (e.g., Dumas (1989), Wang (1996), Chan and Kogan (2002), Garleanu and Panageas (2007)). My paper, however, points out that this familiar mechanism in the literature actually provides a strong argument *against* the commonly held view on the impact of independent biases, an issue that is fundamental to economics and finance. Moreover, one salient feature of my model is about price overshooting: due to biased beliefs, the expected stock return in my model may become negative, while this does not happen in models with only heterogeneity in risk aversion.

Shefrin (2005) and Yan (2008) analyze dynamic models with biased investors with power utility functions and find that due to wealth share fluctuations, biases generally affect the pricing kernel. The current paper shows that the stock price is generally affected by independent biases and derives testable predictions from wealth share fluctuations. It is interesting to note that the fact that biases affect the pricing kernel does not necessarily imply that biases would affect the stock price. For example, if the dividend process is assumed to be the same as the aggregate consumption process, as the literature traditionally does, biases about the dividend growth rate would affect the pricing kernel but not the stock price if all investors have logarithmic preferences (see, e.g., Yan (2008)). Moreover, this paper also demonstrates that biases' impact on the equilibrium depends critically on the variable that investors are biased about.

4 Conclusion

One conventional argument suggests that if biases are independent across investors, they should not have a large impact on equilibrium at the aggregate level since they would cancel each other out. Perhaps partly due to this intuition, the recent behavioral finance literature has made substantial effort to document that investors' biases are often *not* independent and have a significant impact on prices. My paper complements this literature by showing that the effectiveness of this aggregation argument may be more limited than the literature has suggested. In particular, the aggregation argument fails for the following two main reasons. First, if biases affect investor demand in a nonlinear way, they may have a significant impact on the equilibrium even if the biases are independent across investors and the population average belief is unbiased. Second, even if the biases affect investor demand linearly, the aggregation argument may still fail due to the fluctuation of the wealth distribution in a dynamic setting. In particular, an initial run-up of the stock price makes optimistic investors richer, which then further pushes the stock price up and leads to lower future returns. Similarly, an initial drop of the stock price leads to higher future returns. That is, idiosyncratic biases can amplify the fluctuations and induce mean reversion in stock returns. This is in contrast to the effect from the combination of differences in opinion and short sale constraint, which generally implies overvaluation but not undervaluation.

The main theme of this paper is to demonstrate the flaws in the conventional intuition on aggregation. However, the underlying argument can also be taken further to analyze asset price dynamics. For example, Xiong and Yan (2009) show that the impact of independent biases can shed light on a series of stylized facts on the bond price behavior. Moreover, the mean-reversion implication from my model might have contributed to the long-run reversal in stock returns that has been documented in De Bondt and Thaler (1985). This conjecture can be empirically tested if one can measure investors' perceptions from their portfolio holdings.

Appendix

Proof of Proposition 1

Because each investor has $1/N$ share, for a given price P_0, the wealth of each investor is P_0/N. Since investor i allocates a fraction θ_i of his wealth to the stock, his demand for the stock is $P_0\theta_i/N$. Equating the aggregate demand and supply for the stock, we obtain

$$\sum_{i=1}^{N} \frac{P_0}{N}\theta_i = P_0.$$

This implies

$$\frac{1}{N}\sum_{i=1}^{N} \theta_i = 1.$$

When N goes to infinity, the above expression becomes

$$E\left[\theta\left(P_0, \Psi, \tilde{\epsilon}\right)\right] = 1. \tag{31}$$

That is, the equilibrium price in this economy with biased investors satisfies equation (31). Due to the assumptions in (2)–(4), equation (31) has a unique solution, P_0^{**}. The aggregation argument works if and only if $P_0^* = P_0^{**}$, which holds if and only if (8) holds.

A. Proof of Proposition 2

Following Cox and Huang (1989) and Karatzas et al. (1987), market completeness implies that investor i's dynamic budget constraint can be written as a static one, and his optimization problem can be written as

$$\begin{aligned} & \max \ln c_0^i + \beta E_0\left[\ln c_1^i\right] + \beta^2 E_0\left[\ln c_2^i\right] \\ s.t. \qquad & W_0^i = c_0^i + E_0\left[M_1^i c_1^i\right] + E_0\left[M_2^i c_2^i\right], \end{aligned}$$

where c_t^i $(t = 0, 1, 2)$ is investor i's consumption at t, W_0^i is his wealth at $t = 0$, M_1^i and M_2^i are his stochastic discount factors for payoffs at $t = 1$ and $t = 2$, respectively.

Consistency conditions (investors have to agree on the prices of the securities in the market) imply that for $t = 1, 2$,

$$\frac{M_t^0}{M_t^i} = \frac{n^i(\ln D_t)}{n^0(\ln D_t)}, \tag{32}$$

where $n^i(\ln D_t)$ is investor i's probability density function of $\ln D_t$. Applying the standard Lagrange method to solve the maximization problem, we obtain

$$\begin{aligned} c_0^i &= \frac{1}{1+\beta+\beta^2} W_0^i, & (33) \\ c_1^i &= \frac{\beta}{M_1^i} \frac{W_0^i}{1+\beta+\beta^2}, & (34) \\ c_2^i &= \frac{\beta^2}{M_2^i} \frac{W_0^i}{1+\beta+\beta^2}. \end{aligned}$$

Equation (33) shows that the investor's consumption is proportional to his wealth and is not related to his belief. As a result, investor i's consumption share is the same as his wealth share at $t = 0$. By similar argument, this is also true for $t = 1$, and $t = 2$. That is, we have, for $t = 0, 1, 2$,

$$\frac{W_t^i}{W_t^0} = \frac{c_t^i}{c_t^0}. \tag{35}$$

Define

$$\lambda_i \equiv \frac{c_1^i}{c_1^0}. \tag{36}$$

Equation (34) leads to

$$\frac{c_1^i}{c_1^0} = \frac{M_1^0}{M_1^i}. \tag{37}$$

Equations (32), (35), (36) and (37) imply (27), (28) and

$$\lambda_i = \frac{n^i(\ln D_t)}{n^0(\ln D_t)}. \tag{38}$$

Note that (26) directly follows from the assumption that wealth is evenly distributed among investors at $t = 0$.

We can now prove (25). Suppose a security pays K at $t = 1$. Then the price of this security at $t = 0$ can be calculated from investor 0's marginal rate of substitution:

$$S = E^0 \left[\beta \frac{c_0^0}{c_1^0} K \right]. \tag{39}$$

Since all investors have the same wealth share at $t = 0$, they also have the same consumption share. Since the aggregate consumption at $t = 0$ is C_0 (i.e., $C_0 = \sum_{i=-N}^{i=N} c_0^i$), we obtain

$$c_0^0 = \frac{1}{2N+1} C_0. \tag{40}$$

The definition of λ_i in (36) implies

$$c_1^0 = \frac{1}{\sum_{i=-N}^{i=N} \lambda_i} C_1, \tag{41}$$

because the aggregate consumption at $t = 1$ is C_1 (i.e., $C_1 = \sum_{i=-N}^{i=N} c_1^i$). Substituting (40) and (41) into (39), after some algebra, we obtain

$$S = \frac{1}{2N+1} \sum_{i=-N}^{i=N} E^0 \left[\lambda_i \beta \frac{C_0}{C_1} K \right].$$

Substituting (38) into the above expression, we obtain

$$S = \sum_{i=-N}^{i=N} \frac{1}{2N+1} S^i,$$

where $S^i = E^i \left[\beta \frac{C_0}{C_1} K \right]$ is the price of this security in a hypothetical economy with investor i only. By similar arguments, we obtain that the price of *any* security with a finite price can be decomposed into this wealth share-weighted average structure, and (25) is just a special case for the stock price.

B. A continuous-time model

This section presents a continuous-time version of the model in Section 3. This continuous-time model further demonstrates the robustness of the simple two-period model in Section 3.

Consider a pure-exchange economy with a finite horizon $[0, T]$. The exogenous aggregate consumption supply process $C_t > 0$ follows

$$\frac{dC_t}{C_t} = \mu_c dt + \sigma_c dZ_c,$$

where μ_c and σ_c are constants and $\sigma_c > 0$, and Z_c a one-dimensional Brownian motion. The stock is a claim to a dividend process D_t, which follows

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dZ_D,$$

where μ_D and σ_D are constants and $\sigma_D > 0$, $C_0 > 0$, and Z_D a one-dimensional Brownian motion. For simplicity, let's assume dZ_c and dZ_D are independent, and it is straightforward to introduce correlation between dZ_c and dZ_D.

There are $2N+1$ investors who are numbered as $-N, -N+1, ..., -2, -1, 0, 1, 2, ...N$. Investors are biased about the dividend process, but not biased about the aggregate consumption process. More specifically, investors know μ_c, σ_c and σ_D but have different beliefs about μ_D. For $i = -N, ..., N$, investors i's belief μ_D^i is

$$\mu_D^i = \mu_D + \frac{i}{N}\phi. \tag{42}$$

That is, investors' beliefs are evenly spread across $[\mu_D - \phi, \mu_D + \phi]$, and investor 0 has the correct belief. This can be interpreted as that investor i thinks the growth rate is $\mu_D^i = \mu_D + \epsilon_i$, where ϵ_i is an independent realization of a uniformly distributed random variable on $[-\phi, \phi]$. When N is large, investors' biases spread evenly across $[-\phi, \phi]$. Specification (42) is chosen for simplicity. Moreover, it is not essential that investors are not biased about the aggregate consumption. As will become clear, this assumption implies that the biases have no impact on the equilibrium interest rate. Introducing biases about the aggregate consumption would lead to interesting dynamics in the term structure, as analyzed in Xiong and Yan (2009).

From investor i's $(i = -N, ..., N,)$ point of view, the dividend follows

$$\frac{dD_t}{D_t} = \mu_D^i dt + \sigma_D dZ_D^i,$$

where

$$
\begin{aligned}
dZ_D^i &= dZ_D - \delta_i dt, \\
\delta_i &= \frac{\mu_D^i - \mu_D}{\sigma_D},
\end{aligned}
$$

and Z_D^i is a Brownian motion from investor i's point of view. Markets are assumed to be complete: investors can continuously trade a riskless bond and two risky securities. All securities are in zero net supply. The bond price B_t, which is normalized so that $B_0 = 1$, and risky security prices S_{1t}, S_{2t} have the following dynamics:

$$
\begin{aligned}
dB_t &= B_t r_t dt, \\
\frac{dS_{1t}}{S_{1t}} &= \mu_{1t}\, dt + \sigma_{1t}\, dZ_c, \\
\frac{dS_{2t}}{S_{2t}} &= \mu_{2t}\, dt + \sigma_{2t}\, dZ_D.
\end{aligned}
$$

This specific assumption on the financial markets is innocuous. Any three non-redundant securities complete the markets and lead to the same equilibrium.

At time 0, each investor is endowed with $1/(2N+1)$ share of the aggregate consumption. He chooses a nonnegative consumption process c_t^i and holds θ_{jt}^i share of the risky securities S_{jt} (for $j = 1, 2$), and so his financial wealth process W_t^i is

$$
dW_t^i = W_t^i r_t dt - c_t^i dt + \sum_{j=1}^{2} \theta_{jt}^i S_{jt} \left(\mu_{jt} - r_t\right) dt + \theta_{1t}^i S_{1t} \sigma_{1t} dZ_c + \theta_{2t}^i S_{2t} \sigma_{2t} dZ_D. \tag{43}
$$

All investors have the same logarithm preference: investor i's dynamic optimization problem is

$$
\max_{c_i} E_t^i \left[\int_t^T e^{-\rho(s-t)} \log c_s^i ds \right], \tag{44}
$$

subject to the dynamic budget constraint (43), where ρ is the time discount rate and $E_t^i[\cdot]$ is the conditional expectation from investor i's perspective, conditional on the information up to time t.

The definition of equilibrium is standard: a competitive equilibrium is a price system (r_t, S_{1t}, S_{2t}) and each investor's consumption-portfolio processes $(c_t^i, \theta_{1t}, \theta_{2t})$ such that investors choose their optimal consumption-portfolio strategies given their perceived price processes, and good and security markets clear.

Let's first compute the equilibrium in a homogeneous economy. Suppose the economy is populated by investor i only. From the standard consumption asset-pricing formula, the riskless interest rate r_t^i and stock price P_t^i are given by

$$\begin{aligned} r_t^i &= \rho + \mu_c - \sigma_c^2, & (45) \\ P_t^i &= \frac{1 - e^{-\left(\rho - \mu_D^i + \mu_c - \sigma_c^2\right)(T-t)}}{\rho - \mu_D^i + \mu_c - \sigma_c^2} D_t. & (46) \end{aligned}$$

As shown in Basak (2000), in economies with heterogeneous beliefs, the equilibrium can be attained conveniently by constructing a representative investor with stochastic weighting processes, where the weighting processes capture the difference among investors' beliefs. Specifically, define a representative investor with utility function

$$U(c_t; \lambda_t) \equiv \max_{\sum c_t^i = C_t} \sum_{i=-N}^{N} \lambda_t^i e^{-\rho t} \log c_t^i, \tag{47}$$

where $\lambda_t^i > 0$ may be stochastic. With this representative investor formulation, one can easily construct the equilibrium.

Proposition 3 *In the economy described above, the equilibrium stock price is given by*

$$P_t = \sum_{i=-N}^{N} \omega_t^i P_t^i, \tag{48}$$

where ω_t^i is investor i's wealth share, which is the same as his consumption share and is given by

$$\omega_t^i = \frac{\lambda_t^i}{\sum_{j=-N}^{N} \lambda_t^j}, \tag{49}$$

where $\lambda_0^i = 1$ and λ_t^i satisfies

$$d\lambda_t^i = \lambda_t^i \delta_i dZ_D, \tag{50}$$

P_t^i is the stock price that would prevail in an economy populated by investor i only, and is given by (46). The short interest rate is given by (45).

Proof. The following is a brief proof. See Yan (2008) for more details. The λ_t^i process can be determined by investor i's belief and consistency conditions. Since investor 0 has the correct belief, we obtain that λ_t^0 is a constant, and we can normalize it so that $\lambda_t^0 = 1$. Moreover, the construction (47) implies that λ_t^i is the ratio of investor 0's marginal utility to that of investor i. For logarithm preference, it implies $\lambda_t^i = c_t^i/c_t^0$. This, together with the market clearing condition $\sum_{i=-N}^{N} c_t^i = C_t$ and the optimality condition for (44), implies (49) and (50). For logarithmic investors with the same time discount rate, an investor's consumption share is the same as his wealth share. Note that since investors are biased only about the dividend growth, the wealth share is driven only by the dividend shock dZ_D and is independent of the aggregate consumption shock dZ_c. Investors' initial wealth distribution implies that $\lambda_0^i = 1$ for $i = -N, ..., N$. One can compute the stock price by the standard formula

$$P_t = E_t^i \left[\int_t^T e^{-\rho(s-t)} \frac{c_t^i}{c_s^i} D_s ds \right].$$

Substituting c_t^i into the above expression, after some algebra, we obtain (48). Similarly, a bond price can be decomposed into the wealth share-weighted average structure as in (48). However, since there is no bias about the consumption supply process, all investors have the same valuation. As a result, the biases here have no impact on interest rates, and the short rate in this heterogeneous economy is given by (45). ■

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