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FIBONACCI AND THE FINANCIAL REVOLUTION

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Abstract:

This paper examines the contribution of Leonardo of Pisa [Fibonacci] to the history of financial mathematics. Evidence in Leonardo's *Liber Abaci* (1202) suggests that he was the first to develop present value analysis for comparing the economic value of alternative contractual cash flows. He also developed a general method for expressing investment returns, and solved a wide range of complex interest rate problems. The paper argues that his advances in the mathematics of finance were stimulated by the commercial revolution in the Mediterranean during his lifetime, and in turn, his discoveries significantly influenced the evolution of capitalist enterprise and public finance in Europe in the centuries that followed. Fibonacci's discount rates were more culturally influential than his famous series.

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I. Introduction

Present value analysis is a method for comparing the relative economic value of differing payment streams, taking into account the time-value of money. Mathematically reducing all cash flow streams to a single point in time allows the investor to decide which is unambiguously the best. According to a recent survey of corporate financial officers by Graham and Harvey (2001), the present value criterion is now used by virtually all large companies in the capital budgeting decision.[1] The modern present value formula was developed by economist Irving Fisher in 1930, however its origins extend much deeper in the tradition of Western thought.[2] Geoffrey Poitras' excellent survey of the early history of Financial Economics traces related interest-rate problems back to 15th Century Europe.[3] The roots of the present value criterion may lay deeper still, however. In this paper, I argue that an important early formulation of the present value criterion appeared in the work of one of the most famous mathematicians of the Middle Ages, Leonardo of Pisa [1170-1240], commonly called Fibonacci.

Leonardo was arguably the first scholar in world history to develop a detailed and flexible mathematical approach to financial calculation. He was not only a brilliant analyst of the business problems of his day, but also a very early financial engineer whose work played a major role in Europe's distinctive capital market development in the late Middle Ages and the Renaissance. This paper examines the range of Fibonacci's contributions to Finance in historical and economic context, and specifically focuses on his particular development of present value analysis. While many of

[1] Graham, John R., and Campbell Harvey, 2001, "The Theory and Practice of Corporate Finance: Evidence from the Field," *Journal of Financial Economics* 60(2-3), pp. 187-243.
[2] Fisher, Irving, 1930, *The Theory of Interest*, The MacMillan Company, New York.
[3] Poitras, Geoffrey, 2000, *The Early History of Financial Economics, 1478-1776*, Edward Elgar, Northampton.

Fibonacci's mathematical tools -- including his use of interest rates -- were a direct extension of Indian and Arabic mathematics, he appears to have invented a new mathematical approach to financial decision-making. This innovation, in turn, allowed European mathematicians to value increasingly complex financial instruments in the centuries that followed. As such, his intellectual contributions shed some light on the role of mathematics in the broader trajectory of economic development in the Middle Ages and indeed on the distinctive divergence of European society in the world economy.

Fibonacci wrote *Liber Abaci* in the year 1202 in the city of Pisa. It is best known for the Fibonacci series – a sequence of numbers describing geometric population growth. What is less well-known about Fibonacci's book is that the famous series is only a single example in a book almost entirely devoted to the mathematics of trade, valuation and commercial arbitrage. *Liber Abaci* develops a wide range of practical mathematical tools for calculating present value, compounding interest, evaluating geometric series,' dividing profits from business ventures, and pricing goods and monies involving a complex variety weights, measures and currencies. The table of contents gives a sense of the structure and topics covered in the book.[4]

Table of Contents for *Liber Abaci*

1. Here Begins the First Chapter
2. On the Multiplication of Whole Numbers
3. On the Addition of Whole Numbers
4. On the Subtraction of Lesser Numbers from Greater Numbers
5. On the Division of Integral Numbers
6. On the Multiplication of Integral Numbers with Fractions

[4] From Sigler, L.E., 2002, *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*, Springer-Verlag, New York. Spellings of the title of the book differ. I use the spelling from Sigler, but *Liber Abbaci* is also common.

7. On the Addition and Subtraction and Division of Numbers with Fractions and the Reduction of Several Parts to a Single Part
8. On Finding the Value of Merchandise by the Principal Method
9. On the Bartering of Merchandise and Similar Things
10. On Companies and Their Members
11. On the Alloying of Monies
12. Here Begins Chapter 12
13. On the Method Elchataym and How With It Nearly All Problems in Mathematics are Solved
14. On the Finding of Square and Cubic Roots and on the Multiplication, Division, and Subtraction of Them and On the Treatment of Binomials and Apotomes and Their Roots
15. On Pertinent Geometric Rules and On Problems of Algebra and Amuchabala

Liber Abaci begins with an exposition of the fundamentals of arithmetic, followed by chapters on valuation, relative value, companies and metallurgy. Chapter 12 contains a wide variety of problems, including interest rate and present value examples. Chapter 13 presents a method of solution through linear interpolation, and Chapters 14 and 15 present more abstract mathematical results. Proceeding as it does from the simple to the complex, the book is organized as a manual for instruction, and it is likely that Leonardo used it himself for teaching commercial arithmetic.

However, it is not only the content and use of the book that makes it important, but the historical and economic context. *Liber Abaci* was written at a time of nearly unprecedented commercial interaction between East and West. Italian merchants traded actively with the Arab world – bringing exotic goods to an increasingly prosperous Europe. This period of economic vitality stimulated the development of practical mathematics – in effect, creating a market for mathematical knowledge. This knowledge, in turn became a major factor in the economic trajectory of Europe over the next several centuries. The 500 year period following Leonardo saw the development in Europe of virtually all the tools of financial capitalism that we know today: share ownership of limited-liability corporations, long-term government and corporate loans, liquid and active international financial markets, life insurance, life-annuities, mutual funds, derivative securities and deposit banking. Many of these developments have their roots in the contracts first mathematically analyzed by Fibonacci. What began with his application of mathematics to current commercial problems may have led to important later economic innovations and the development of Western capitalism as we know it today.

East and West

Leonardo's contribution is also interesting in a global context. Historians have long puzzled over the extraordinary economic divergence between Western Europe and China over the period from 1200 to 1700 AD.[5] These innovations are all the more puzzling in the context of China's comparatively advanced scientific, technological, political and cultural advantages at the time.[6] For example, Marco Polo's account of China in the late 13th Century highlighted the extraordinary advancement of Chinese civilization in Western eyes. Polo and his fellow travelers encountered a vast nation politically unified under Mongol rule. The wonders of Yuan China included magnificent cities, a widely literate population and highly advanced art, science and technology.

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[5] See, for example, Kenneth Pomeranz, 2000, *The Great Divergence: China, Europe, and the Making of the Modern World Economy*, Princeton University Press, Princeton.

[6] See the on-going publication of Needham, Joseph, 1954-, *Science and Civilisation in China*, Cambridge University Press, Cambridge -- a work that definitively documents the level of scientific advancement of China, and indeed gave rise to the so called 'Needham Puzzle" -- see for example, Lin , Justin, 1995, "The Needham Puzzle: Why the Industrial Revolution Did Not Originate in China, " *Economic Development and Cultural Change*, 43(2) pp. 269-292.

More relevant to our study, however, China also had a sophisticated system of government finance that included both a paper and a metallic currency, a system of national government monopolies on commodities, closely regulated commercial markets, as well as a centuries-long legal tradition of property rights. By contrast, Europe of Polo's day was a fragmented patchwork of cities, states and communes, only partially unified by the holy Roman empire and the Crusades. The Mediterranean world was politically fragmented in the Middle Ages but economically connected by a network of commerce. Thus, almost by definition, virtually all trade in Europe in the 13^{th} century was international.

The Mongol rulers of Macro Polo's era in the Khan's capital near modern-day Beijing could sip tea picked thousands of miles away in China's southern province of Yunan and shipped through China – perhaps even via the great man-made inland canal system linking the Yangtse and Yellow River valleys. They could read books published in the distant Southeastern Chinese province of Fujian, paid for in national currency accepted across the breadth of China. On the other hand, a merchant in Fibonacci's Pisa who traded goods just within Northern Italy had to be able to calculate currency conversions from the money of each city-state – mastering the comparative values and differential silver content of Bolognese, Pisan, Venetian and Genoese lira, and knowing their value relative to Byzantine besants, Imperial pounds, Barcelona lira and Magalonese soldi. The very complexity of European commerce at the time necessitated arithmetical tools for solving problems of conversion and exchange. Fibonacci's book thus met an important commercial need.

Perhaps more relevant to the issue of the great divergence between East and West however is that Fibonacci's mathematical technology in general, and the present value criterion in particular, facilitated the development of financial contracts, instruments and markets in Europe. Some of these financial contracts existed in Europe or were created during Fibonacci's lifetime, however the mathematical capacity to calculate the present value of differing instruments led to a sustained period of financial innovation in a relatively localized geographical area. The financial divergence that began in 13th Century Italy culminated most dramatically in the commercial encounter between China and the great capitalist powers of the 19th Century, with the opening of the treaty ports of Canton and Shanghai after the Opium Wars. At that moment, a country with a long tradition of family and clan-based finance met a culture of business armed with the extraordinary capacity to borrow and raise equity capital from complete strangers. This encounter offered both a challenge and an opportunity to Chinese entrepreneurs and government officials and led to a rapid process of economic and political change that continues into the modern era.

II. Fibonacci and the Commercial World of the 13th Century

Interest in the Medieval origins of the European commercial revolution has stimulated some remarkable scholarship – from the extensive archival explorations into the roots of capitalism by the late Robert Lopez and his students to the geography-based quantitative traditions of the late Ferdinand Braudel and his followers. The contributions of both narrative and quantitative history has created a rich picture of the sudden emergence of trade and business in Medieval Europe after the year 1000. Most characterize it -- for better or worse -- as a commercial revolution and an economic awakening from the Dark Ages. Although the scholarly contributions based on primary archival sources are too numerous to cite here, synthetic works in this tradition include Ferdinand Braudel's classic three-volume work, *Civilization and Capitalism*, Joseph Gies' *Merchants and Moneymen: The Commercial Revolution, 1000-1500*, and Robert S. Lopez' *The Commercial Revolution of the Middle Ages, 950-1350.* [7] The dramatic change during this period has recently stimulated economists to analyze the institutional structures that supported the new long-distance trade. Avner Greif, for example, suggests that Italian cites had to develop new ways of dealing with foreign merchants – including internally and externally recognized coalitions and business associations.[8] In other recent work, Peter Spufford describes the world of the merchant in 13th century Medieval Europe in detail and points out that the very factors that impeded European trade – from bad roads and a multiplicity of currencies, to varying legal standards and lack of credit markets also created opportunities for entrepreneurs who could surmount them.[9] Wealthy urban centers throughout the continent stimulated a demand for exotic goods and an enterprising merchant and banking sector emerged to meet this demand – extending mercantile ties outside of Europe to the wider basin of the Mediterranean and beyond, importing goods from the Middle East and occasionally even further afield. Marco Polo and his brothers, for example, manned a Venetian trading station in the

[7] See, for example, Braudel, Ferdinand, 1973-1984, *Civilization and Capitalism 15th-18th Century* 3 vols. University of California Press, Berkeley. Gies, Joseph, 1972, *Merchants and Moneymen: The Commercial Revolution, 1000-1500*, Ty Crowell, New York, and Lopez, Robert S. 1971, *The Commercial Revolution of the Middle Ages, 950-1350*, Prentice-Hall, Englewood Cliffs.
[8] Greif, Avner, 1989, "Reputation and Coalitions in Medieval trade: Evidence on the Maghribi Traders," *The Journal of Economic History*, 49(4), December, pp. 857-882.
[9] Spufford, Peter, 2002, *Power and Profit: The Merchant in Medieval Europe*, Thames and Hudson, New York.

region of the Black Sea from which he either launched his long journey to China, or perhaps simply collected travelers accounts of distant Cathay.

Fibonacci and his family were also an integral part of this commercial revolution.[10] According to Leonardo's brief autobiography in *Liber Abaci*, his father was an administrative official in the Pisan colony of Bugia in North Africa. Leonardo was summoned as a youth to the colony, and trained there according to his father's request in Arabic mathematical methods. As a result of Fibonacci's bi-cultural education, he enthusiastically embraced "the Indian Method" of calculation, i.e. the use of Arabic numerals. In fact, while the use of Arabic numerals had appeared sporadically in Western Europe before Fibonacci's time, *Liber Abaci* was one of the first works to methodically describe the new number system to Europeans and to demonstrate its practical, commercial use through detailed and copious examples.

The Pisan colony of Bugia in the Middle Ages was an important source for two trade items: a fine grade of beeswax and high-quality leather. Originally a minor Roman colony, the city was revived in the 11th Century by the Berbers. The Pisans took the Algerian colony from their Genoese rivals in the mid-12th Century, and established a trading port that not only exported North African goods, but also served as a conduit of Eastern luxury items of all kinds into Europe.

Leonardo's father served as a customs official and perhaps local representative of the merchants in the Pisan colony in the late-1100's and thus may have had

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[10] Gies, Joseph and Frances, 1969, *Leonardo of Pisa and the New Mathematics of the Middle Ages*, Harper-Collins, New York , provide a brief, but very readable popular account of the life of Leonardo.

responsibility for recording the inflow and outflow of the goods through the port.[11] At least two problems in *Liber Abaci* concern the calculation and payment of customs duties. Indeed, as an official representative of the Italian merchants in Bugia, Leonardo's father would presumably have understood the nature and value of the range of merchandise in which they traded, and perhaps even the system of record and notation, accounts and language used by Arab traders. Leiber (1968) notes regular business correspondence in Arabic between Muslim and Pisan merchants from the years 1200 to 1202.[12] Thus, Leonardo not only had an education in Eastern mathematics, but deep family roots in Mediterranean trade.

In his own account of his life, Leonardo explains that he traveled extensively throughout the Mediterranean as a young man – presumably as a merchant himself. He notes that he pursued mathematical knowledge from "whoever was learned in it, from nearby Egypt, Syria, Greece, Sicily and Provence …to which locations of business I traveled afterwards for much study." [13] Leonardo returned from his travels sometime around 1200, at about age 30 and published *Liber Abaci* in 1202. He revised it in a second edition of 1228. The oldest extant edition of the book is a handwritten manuscript from 1291, now in the Biblioteca Riccardiana in Firenze.[14] Copied more than

[11] See Leiber, Alfred E., 1968, "Eastern Business Practices and Medieval European Commerce," *The Economic History Review*, New Series, 21(2) August, pp. 230-243. According to Lieber, European merchants who traded with eastern Mediterranean ports may have exported regulatory practices as well as goods. The modern French word "duanes," for example, derives from the Arabic "dowanes" for customs.
[12] Ibid.
[13] Sigler, (2002) Op.cit. p.15. It of interest that Leonardo learned mathematics from teachers, as opposed to books. Although he cites other works, his own account suggests that mathematical education during his lifetime was, at least in large part, an oral tradition.
[14] This is the edition translated into modern Italian by Baldassare Boncompagni, *Scritti di Leonardo Pisano, matematico del secolo decimoterzo, vol. 1 : Il Liber Abaci di Leonardo Pisano*, pubblicato secondo la lezione del codice Magliabechiano C. I, 2616, Badia Fiorentina, n° 73, Tipografia delle scienze matematiche e fisiche, Rome, 1857. A useful description of the manuscript can be founding Van Edgemond, Warren, 1980, *Practical Mathematics in the Italian Renaissance: A Catalogue of Italian*

sixty years after the original, according to Siegler (2002) it contains a variety of minor mathematical errors – some noted by the later copyist, other not, suggesting a slow accretion of mistakes through repeated hand-copying over the decades from 1228 to 1291 – implying among other things a sustained interest in the book after its original appearance.[15] Indeed, the fact that Leonardo produced a second edition attests to the interest in the book even during his lifetime.

Little else is known of Leonardo's life in Pisa, except that he wrote a number of other mathematical treatises, and corresponded with members of Emperor Fredrick II's court. The emperor himself is thought to have read *Liber Abaci*, and, on one of his visits to Italy, granted the famous mathematician an audience.

Leonardo received a pension from the Republic of Pisa in 1241 for " educating its citizens and for his painstaking, dedicated service." Professor Heinz Lüneburg has posted a Latin transcript of the document – it suggests that the city was equally grateful for his teaching (*doctrinum*) and his accounting and valuation (*abbacandis estimationibus et rationibus*).[16] We can thus surmise from this that much of his life was devoted to teaching mathematics and advising the Pisan government on issues of finance and accounting.

Abbacus Manuscripts and Printed Books to 1600. Supplemento agli *Annali dell'Instituto e Museo di Storia della Scienza*, Fascicolo 1, Firenze.

[15] Sigler (2002), Op.cit. p. 10.

[16] "…qui eis tam per doctrinam quam per sedula obsequia discreti et sapientis viri magistri Leonardi Bigolli, in abbacandis estimationibus et rationibus civitatis …" http://www.mathematik.uni-kl.de/~luene/miszellen/Fibonacci.html Accessed by author 10/25/2003.

Sources and Predecessors

Leonardo gives considerable credit throughout *Liber Abaci* to the work of Muhammad ibn Mūsá Al Khwārizmī [c.780-850], the 9th century court astronomer of Baghdad who wrote the famous treatise *The Algebra*, for whom that entire branch of mathematics is named.[17] A few earlier "Algorisms" – as translations and variations on *The Algebra* are called -- appear in European context before the publication of *Liber Abaci*. The earliest dates to the early or Mid-12th century.[18] However these other algorisms hone fairly closely to Khwārizmī's work.

Liber Abaci, obviously owes an intellectual debt to Al Khwārizmī, not only in his use of Indian numerals but also in the extensive use of algebraic methods. The two works differ significantly in many ways, however. *Liber Abaci* is a much longer book, containing many more problems and solutions and introducing a considerably broader range of practical applications. More importantly, the two authors treat fundamentally different types of practical problems. For example, most of Al Khwārizmī's non-geometric examples are problems of inheritance. They deal with the complexity of dividing assets among family members and other claimants. A typical question is:

> [A man] leaves two sons and ten dirhems of capital and a demand of ten dirhems against one of the sons, and bequeaths one-fifth of his property and one dirhem to a stranger.."[19]

[17] An English translation of *The Algebra* is Rosen, Frederic, 1831, *the Algebra of Mohammed Ben Musa*, the Oriental Translation Fund and J. Murray, London.

[18] For an account of the earliest European algebras, see Menninger, Karl, 1969, *Number Word and Number Symbols*, MIT Press, Cambridge, pp. 410-412. Also useful is Karpinski, Louis C., 1921, "Two Twelfth Century Algorisms" *Isis*, 3(3), Summer, pp. 396-413 and Benedict, Suzan Rose, 1914, *A Comparative Study of the Earliest Treatises Introducing Into Europe The Hindu Art of Reckoning*, Thesis submitted to the University of Michigan.

[19] Rosen (1831) Op.cit. p. 87.

Thus, Al Khwārizmī's focus is not fundamentally commercial, but directed towards legal issues related to legacies and dowries – a class of problems that might in fact have inspired him to write his book. As we will see in further discussion of *Liber Abaci* below, Leonardo takes the solution methods from Al Khwārizmī's *Algebra* and adapts them to problems of dividing the capital from commercial ventures among partners, rather than among family members. Thus, the works share methodological approaches but differ in applications and scope.

An equally important precedent for *Liber Abaci*'s more commercial orientation is the practical tradition of Hindu mathematics. For at least seven centuries before Fibonacci, Indian mathematicians were calculating interest rates and investment growth. For example, one of India's earliest mathematicians, Āryabhata [475 – 550] presents and solves some interest rate problems in his famous book, *Āryabhatīya* -- a work otherwise known for its contribution to astronomy.[20] In the 7th Century, Bhāskara [c. 600-680] wrote and extension and commentary on *Āryabhatīya* which contains several practical applications of the mathematics in the earlier work, including partnership share divisions, and the relative pricing of commodies.[21] Sridharacarya's [c. 870-930] 10th Century *Trisastika*, a work of 300 verse couplets, contains a few very practical interest rate problems and a division of partnership problem.[22]

One particularly interesting example of an Indian mathematical precedent to Leonardo's work is a problem posed by the ninth-century Jain mathematician Mahavira

[20] Clark, Walter E., 1930, *The Āryabhatīya of Āryabhata*, The University of Chicago Press, Chicago, pp. 38- 40. Clark discusses the possibility that some portions of the work were actually added substantially later – perhaps as late as the 10th Century.

[21] Sarma, Sreeramula Rajeswara, 2002, "Rule of Three and its Variations in India," Dold-Samplonius et. a. eds. *From China to Paris: 2000 Years Transmission of Mathematical Ideas,"* Franz Steiner Verlag, Stuttgart pp. 133-156.

[22] Ramanujacharia, R and Kaye, G.R., 1913, "The Trisatika of Sridharacarya," *Bibliotheca Mathematica*, series 3, 1913, pp. 203-217.

[c. 800-870]in his *Ganita Sara Sangraha.* In the Indian work, three merchants find a purse lying in the road. The first asserts that the discovery would make him twice as wealthy as the other two combined. The second claims his wealth would triple if he kept the purse, and the third claims his wealth would increase five fold. Precisely the same problem appears in Chapter 12 of *Liber Abaci*, leaving little doubt that at least certain classical problems in Leonardo's book derive from Indian texts – whether transmitted by word of mouth over centuries through the Arab world, or even directly through now-lost Arabic translations of Hindu works. Closer to Leonardo's era, and very close to the spirit of the financial problems in the Pisan's work, the *Lilivati* of Bhāskarācārya [1114-1185], dates to about 1150 A.D. and, like the earlier works *Trisastika* and *Āryabhatīya,* contains some loan problems and methods of finding principal and interest.[23] Despite broad parallels, however, none of the earlier Indian works develop a present-value criterion.

Even in Europe there may have been mathematicians who shared Fibonacci's interest in applied mathematics. Savasorda [d. 1136], a Jewish encyclopoedist in 12th Century Barcelona composed a mathematical treatise on arithmetic and geometry, but also solved problems involving price, quantity and money.[24] Levey (1952) argues from close textual similarities that Fibonacci must have seen Savasorda's work and adapted his methodology of proof, although the work is not cited by the Pisan. Perhaps a follower

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[23] The legal tradition in India pertaining to loans and interest rates is a long one. See Kane, M.P.V., 1946 *History of Dharmaśastra* , Government Oriental Series Class B Ro. 6, Bhandarkar Oriental Research Institute, Poona, vol. 3 p. 419 and ff. This classic study of India's ancient and medieval law describes legal texts dating from the first four centuries A.D. and deals in detail with the classification and calculation of interest on loans of various sorts, from unsecured debt to mortgages to investment in seafaring enterprises. Thus the commercial context for the development of the mathematics of discounting in India preceded the surviving mathematical texts on the subject.

[24] Levey, Martin, 1952, "The Encyclopedia of Abraham Savasorda: A Departure in Mathematical Methodology," *Isis*, 43(3) September, pp. 257-264.

of Savasorda was one of the unnamed mathematics masters whom Fibonacci visited on his travels.

Thus, although we will never know all of Fibonacci's sources for mathematical methods and problems, his acknowledged Arabic precedents, and the likely multiple Indian precedents, provide at least a conceptual foundation for his work. Never-the-less, the differences are as interesting as the similarities. No previous work of algebra or applied mathematics comes close to the comprehensive analysis of business and financial transactions of *Liber Abaci*, and none provides a comparable text for the actual teaching of applied mathematics. While it is structured a bit like previous algorisms – with the number system first, followed by the introduction of basic operations and then fractions, a large part of the book is devoted to progressively more complex word problems. In chapter after chapter, Fibonacci first introduces a very simple problem and its solution, and then provides increasingly sophisticated examples that use the same method of solution and require applications of tools mastered in previous chapters. Let us now turn to some of *Liber Abaci*'s key contributions.

Fractions and The Rule of Three

After an initial discussion of arithmetic operations in the first four chapters, chapters five and six of *Liber Abaci* introduce fractions and mixed numbers. Leonardo's fractions are a bit different from the modern form. They are "factored." For example the number 5.123 would be written $\frac{3}{10}\frac{2}{10}\frac{1}{10}5$. That is, the denominator of the leftmost fraction is the product of all denominator values that precede it. This method was particularly useful for calculations involving non-decimal monetary and quantity systems. For example, monetary units based upon the Roman system used denari, soldi, and lira, with 12 denari to the soldo and 20 soldi to the lira. In this notation, the units of each denomination can be preserved in the fraction -- a price of five lira, six soldi and four denari can be represented as $\frac{4}{12}\frac{6}{20}5$. Units of weight and measure can be even more complex. According to Leonardo,

> Pisan hundredweights … have in themselves one hundred parts each of which is called a roll, and each roll contains 12 ounces, and each of which weighs ½ 39 pennyweights; and each pennyweight contains 6 carobs and a carob is 4 grains of corn.[25]

Chapters 8 is entitled "on Finding the Value of Merchandise by the Principal Method," and it begins the application of the new mathematics to commercial problems. The first problem he addresses is how to determine an unknown price from a given quantity of merchandise when the price per unit is known -- suppose a 100 rolls costs 40 lira, how much would five rolls cost? He offers the solution through a diagram $\begin{matrix} 40 & 100 \\ ? & 5 \end{matrix}$ and the solution as (40 * 5)/100.

This simple solution is generally called "The Rule of Three," and is one of the oldest algebraic tools in mathematics. The Rule of Three appears in the *Āryabhatīya*, and is extended and elaborated upon in Bhāskara's commentaries, in which he applies it to problems quite similar to those analyzed by Leonardo.[26]

Leonardo uses the Rule of Three with increasingly complex quantities and currencies, applying it to examples drawn from trade around the Mediterranean. Goods

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[25] Sigler (2002), Op.cit. p. 128.
[26] See Sarma (2002) Op. cit. for an early history of the Rule of Three.

include hundredweights of hides, hundredpounds of pepper, tons of Pisan cheese, rools of saffron, nutmeg and cinnamon, meters of oil, sestario of corn, canes of cloth, oil from Constantinople. Currencies include denari, massamutini, bezants, tareni. Leonardo colorfully describes transactions in Sicily, the Barbary Coast, Syria, Alexandria, Florence, Genoa, Messina and Barcelona. The prosaic nature of each of these problems speaks of the practical import and the likely appeal of the book to merchants. For example, for a Pisan cloth merchant who trades perhaps in Syrian damask, and competes with a Genoese merchant, it would have been useful to be able to translate across three different units of length:

> A Pisan cane is 10 palms or four arms however a Genoese cane is said to be 9 palms. And furthermore the canes of Provence and Sicily and Syria and Constantinople are the same measure.[27]

Or for traders of raw cotton here is a useful problem:

> One has near Sicily a certain ship laden with 11 hundreweights and 47 rolls of cotton, and one wishes to convert them to packs; because 1/3 1 hundredweights of cotton … is one pack, then four hundredweight of cotton are 3 packs and for rolls of cotton are 3 rolls of a pack; you write down in the problem the 11 hundredweights and 47 rolls , that is 1147 rolls below the 4 rolls of cotton and you will multiply the 1147 by three and you divide by the 4; the quotient will be ¼ 860 rolls of a pack.[28]

Following this basic technology, Fibonacci then extends the problems to those of exchange between two goods – for example:

> It is proposed that 7 rolls of pepper are worth 4 bezants and 9 pounds of saffron are worth 11 bezants, and

[27] Sigler (2002) Op.cit. . p.168.
[28] Ibid. p. 176.

it is sought how much saffron will be had for 23 rolls of pepper.[29]

Here, the diagram is expanded to three columns. $\begin{bmatrix} \textit{saffron} & \textit{bezants} & \textit{pepper} \\ \left[\frac{2}{7}\,\frac{8}{11}10\right] & 4 & 7 \\ 9 & 11 & 23 \end{bmatrix}$[30]

Showing how to multiply 23 * 4 * 9 and divide by the diagonal elements 7*11 to arrive at the solution. This pattern can thus be applied to any longer sequence of intermediate trades to establish no-arbitrage relationships among commodities in a market. This solution is essentially the application of the "Rule of Five" that Bhāskara and later Indian mathematicians developed for expressing price/quantity relationships across several goods. Sarma (2002) points out that the horizontal expression of The Rule of Five in Western algorisms is orthogonal to the vertical orientation of the Indian application of the rule.[31] Thus, either an intermediate Arabic manuscript changed the algorithm, or the solution method was transmitted West conceptually, without a written Indian model.

Although the mathematics of the Rules of Three and Five are trivial to us 800 years later, they were unquestionably a vital quantitative tool of the arbitrageur of the 13th Century. Italian merchants were buying saffron and pepper from their Arab counter-parties, who were the primary intermediaries in the spice trade. Merchants who could not calculate the relative value of saffron and pepper in the market – or perhaps could only do so approximately, or with some difficulty -- were at an extreme disadvantage in trade and negotiation. Just as today's hedge funds use sophisticated, quantitative models to rapidly calculate the relative price of two mortgage-backed securities, and then use these calculations to establish a long or a short position if they detect a mispricing, so must 13th

[29] Ibid. p. 184.
[30] Ibid.
[31] Sarma (2002) Op. cit.

Century Pisan merchants, bartering in a Damascus *suk* with spice traders, have used a rapid series of arithmetic calculations to detect and exploit deviations from price parity and barter their way towards profit. It is thus not surprising that the knowledge of the Rule of Three extended in antiquity along trade routes.

Foreign Exchange and Coinage

Leonardo also applies the Rules of Three and Five to problems of currency exchange. Here, the practical demand for a guide to monetary conversions must have been considerable. Peter Spufford charts the evolution of an extraordinary patchwork of currencies in Medieval Europe around Leonardo's time [See Figure 1].[32] Italy had the highest concentration of different currencies with 28 different cites at one time or other issuing coins in the Middle Ages -- 7 in Tuscany alone. Most, but not all, of them were based on the Roman system of "*d,s,l*" denari, soldi, lire – familiar to English readers as the pounds, shilling and pence system. However their relative value and metallic composition varied considerably through time and across space. The varieties of currency created business for money changers, "Banche Del Giro," where monetary systems came into contact, and these money changers and their customers needed tools for calculation. Before Fibonacci, the common method for exchange arithmetic was the abacus, or versions based on moving markers in grooves or along scored lines on a table. As such, the algorithmic solution to an exchange problem was separated from the written expression of the problem – Arabic numerals allowed the solution to be calculated on paper based upon the written statement of the problem. This almost certainly altered and

[32] Spufford, Peter, 1986, *Handbook of Medieval Exchange*, Offices of the Royal Historical Society, London.

improved the way money changers could model and analyze relative values of multiple currencies rapidly. It is one thing to be able to accurately calculate a sequence of monetary exchanges one at a time – it is another thing to be able to set up a series of equations of relative value, and to place algebraic bounds on how much any currency is worth in terms of another. Reading *Liber Abaci* one has the sense that Italian merchants of the 13th century operated in a world of complete relativism. With no central government, no dominant currency, and even competing faiths and heresies, value is expressed quite abstractly only in a set of relative relations to other items. While many of Leonardo's examples use money as a unit of calculation, the currencies vary arbitrarily from example to example – almost as if to stress the flexibility and broad applicability of his algebraic approach.

Indeed, money of his day was not quite money in the way we currently think of it. Transactions might be expressed in a certain currency, or unit of account, but the quantity of physical coinage corresponding to that unit could vary through time, as governments regularly debased and re-valued their currencies. One of the attractive things about a pure gold coin, like the Florentine florin first minted in 1252, is that it could serve simultaneously as a unit of account and of transaction, with a 1:1 numeric correspondence. Most silver coinage of the time had at least some level of debasement with copper that caused its value relative to bullion to fluctuate through time. It is thus not surprising that Leonardo's monetary analysis goes beyond currency conversions into problems of minting and alloying of money. An entire chapter is devoted to the methods for minting coins of silver and copper.

Certain cities – including Pisa -- had imperial rights to coin money, and mints in these cities operated by striking coins from bullion brought to them from the public and the private sector. A merchant who wished to turn a quantity of metal (perhaps even old coins) into currency would pay a seigniorage fee to the mint for the production of coin. During much of Leonardo's lifetime, the Pisan penny was pegged to the value of the Lucchese penny and together they became the standard monetary unit in Tuscany, with little evidence of debasement or changes in the metallic composition. In contrast, also during Leonardo's lifetime, the currency of Northern Italy underwent a revolution -- cites began to mint large silver coins called *grossi*. In the late 1220's (around the time of the second edition of *Liber Abaci*) Lucca introduced a coin with a high silver content worth 12 denari.[33] This followed the introduction of *grossi* by Genoa in 1172 and by Venice in 1192. Leonardo does not specifically address the relative value of *grossi* and *denari* on his chapter on minting of coin, instead abstracting away from particular currencies to the general methods of calculation and evaluation of relative proportions of silver and copper in coins. Presumably the methodology developed in *Liber Abaci* was useful not only to the master of the Pisa mint, but perhaps to merchants who brought metal to be coined, and paid the seignorage.

III. Finance and Interest Rate Problems

Most of the pure finance problems in *Liber Abaci* are in the twelfth chapter – as is the famous "Fibonacci series" example. The finance in this chapter is of four general types. The first type concerns how the profits from joint business ventures should be

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[33] Velde, François R, 2000, "Notes on Tuscan Coinage, 12th C. 14th Ce." Federal Reserve Bank of Chicago.

fairly split when contributions are unequal and are made at different points in time, and in different currencies or goods and in cases in which business partners borrow from each other. The second type concerns the calculation of the profits from a sequence of business trips in which profit and expense or withdrawal of capital occurs at each stop. The third is the calculation of future values from investments made with banking houses. The fourth is present value analysis as we now think of it, including specifically the difference between annual and quarterly compound interest.

Division of Profits

The mathematics of dividing profits from business joint ventures was obviously relevant to Italian merchants of the 13th Century. The basic business unit used to finance many of the trade ventures in Northern Italy in the Middle Ages was the *commenda* contract between an investor and his traveling partner – the former (*commedator*) investing capital and the latter (*tractator*) investing labor. John H. Pryor's (1977) study of *commeda* from the 12th Century describes two basic types – a unilateral and a bi-lateral contract – the former providing limited liability but smaller profit to the *tractator* and the latter sharing the potential losses equally between the contracting parties.[34] In the standard, uni-lateral commenda, the *commendator* would transfer capital to the *tractator* for the duration of the voyage (or term of the contract) and would take ¾ of the profit. Much of what is known about *commeda* has been gleaned from studies of notarial documents in Italian and French archives, and Pisan records are among the oldest. Pisa's *Constitutum Usus* (1156) is the earliest surviving municipal document specifying the

[34] Pryor, John, H., 1977, "The Origins of the Commenda Contract," *Speculum*, 52(1) January, pp. 5-37.

conditions of the *commeda* contract.[35] The unilateral *commeda* in Pisa was very much like the contract described by Leonardo in a problem of division of profits, writing 44 years after the statues. The problem "On Companies," is worth quoting at length:

> Whenever … any profit of an association is divided among its members we must show how the same must be done according to the abovewritten method of negotiation …We then propose this of a certain company which has in its association 152 pounds, for which the profit is 56 pounds, and is sought how much of the same profit each of its members must be paid in pounds. First, indeed, according to Pisan custom, we must put aside one fourth of the abovewritten profit [apparently for the *tractator*] ; after this is dealt with, there remain 42 pounds. Truly if you wish to find this according to the popular method, you will find the rule of 152, that is $\frac{1\;0}{8\;19}$; you divide the profit, namely the 42 pounds; by 8; the quotient will be 5 pounds and 5 soldi which are 105 soldi, which you divide by the 19; the quotient will be 5 soldi and $\frac{6}{19}$ 6 denari. $\begin{array}{cc} \textit{profit} & \textit{capital} \\ \end{array} \begin{bmatrix} 42 & 152 \\ \frac{6}{19}\frac{6}{12}\frac{5}{20} & 1 \end{bmatrix}$ If you wish to find by the aforewritten rule how much will result … for profit from 13 pounds in the association, then you do this: multiply the 13 by the profit prortion in one pound, namely by 5 soldi and $\frac{6}{19}$ 6 denari.. there will be 3 pounds and 11 soldi and $\frac{2}{19}$ 10 denari. $\begin{array}{cc} \textit{profit} & \textit{capital} \\ \end{array} \begin{bmatrix} 42 & 152 \\ \frac{2}{19}\frac{10}{12}\frac{11}{20}3 & 13 \end{bmatrix}$.[36]

Fibonacci's problem provides a fascinating example of how *commenda* were used. Although the *tractator* profits were divided according to Pisan custom, the *commendator* shares could be divided among several investors, whose association is termed a *societas*. Fibonacci thus develops a general approach to dividing the profits from a *societas*

[35] Sigler (2002) Op.cit. p.10.
[36] Sigler (2002) p. 172.

according to the proportion of contributed capital. His approach is the now familiar definition of a rate of return on a unit of capital which is then scaled by the amount invested. Robert Lopez argues that *societas* and the *commenda* contracts were the original European forms of business association from which modern partnership and corporations ultimately developed. Consequently, the *commenda* example, and related problems addressed by Fibonacci in *Liber Abaci,* were significant early contributions to "pre-corporate" financial economics.

Traveling Merchant Problems

The second type of financial problem is a set of "traveling merchant" examples, akin to accounting calculations for profits obtained in a series of trips to trading cities. The first example is:

> A certain man proceeded to Lucca on business to make a profit doubled his money, and he spent there 12 denari. He then left and went through Florence; he there doubled his money, and spent 12 denari. Then he returned to Pisa, doubled his money and it is proposed that he had nothing left. It is sought how much he had at the beginning.[37]

Leonardo proposes an ingenious solution method. Since capital doubles at each stop, the discount factor for the third cash flow (in Pisa) is ½ ½ ½ . He multiplies the periodic cash flow of 12 denari times a discount factor that is the sum of the individual discount factors for each trip i.e. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$. The solution is 10½ denari. The discount factor effectively reduces the individual cash flows back to the point before the man reached Lucca. Notice that this approach can be generalized to allow for different cash flows at

[37] Sigler (2002) Op.cit. p. 372.

different stages of the trip, a longer sequence of trips, different rates of return at each stop, or a terminal cash flow. In the twenty examples that follow the Lucca-Florence-Pisa problem, Leonardo presents and solves increasingly complex versions with various unknown elements. For example, one version of the problem specifies the beginning value and requires that the number of trips to be found – e.g. "A certain man had 13 bezants, and with it made trips, I know not how many, and in each trip he made double and he spent 14 bezants. It is sought how many were his trips."[38] This and other problems demonstrate the versatility of his discounting method. They also provide a framework for the explicit introduction of the dimension of time, and the foundation for what we now consider finance.

Interest Rate and Banking Problems

Immediately following the trip problems, Fibonacci poses and solves a series of banking problems. Each of these follows the pattern established by the trips example – the capital increases by some percentage at each stage, and some amount is deducted. For example:

> A man placed 100 pounds at a certain [banking] house for 4 denari per pound per month interest and he took back each year a payment of 30 pounds. One must compute in each year the 30 pounds reduction of capital and the profit on the said 30 pounds. It is sought how many years, months, days and hours he will hold money in the house....[39]

Fibonacci explains that the solution is found by using the same techniques developed in the trips section. Intervals of time replace the sequence of towns visited and thus a time-

[38] Ibid. p. 383.
[39] Ibid. p. 384.

series of returns and cash draw-downs can be evaluated. Once the method of trips has been mastered, then it is straightforward to construct a multiperiod discount factor and apply it to the periodic payment of 30 pounds – although in this problem the trick is to determine the number of time periods used to construct the factor. Now we might use logarithms to address the problem of the nth root for an unknown n, but Fibonacci lived long before the invention of logarithms. Instead, he solves it by brute force over the space of three pages, working forward from one period to two periods etc. until he finds the answer of 6 years, 8 days and $\frac{1}{2}\frac{3}{9}$ 5 hours. The level of sophistication represented by this problem alone is unmatched in the history of financial analysis. Although the mathematics of interest rates had a 3,000 year history before Fibonacci, his remarkable exposition and development of multi-period discounting is a quantum leap above his predecessors.

Eleven banking house problems follow this initial mathematical *tour de force*. Each of them has some value of the problem that is unknown – either the future value, the present value, the rate of interest, the duration of the loan, or the periodic dividend that is withdrawn. Fibonacci provides model solutions to each unknown, basing all of them on the previously introduced "trips" paradigm.

It is a bit surprising to find such an overt description of the charging of interest by bankers in the 13^{th} Century, given the ecclesiastical proscription against usury. John Munro has argued convincingly that, while Church doctrine had long forbidden usury, vigorous institutional attacks on the charging of interest began in earnest in the early 13^{th} century and were a major impetus for the development of *census* and *rentes* and perpetual bond contracts – all of which circumvented the ecclesiastical definition of loans.[40] *Liber Abaci* was written just before the formation of the Franciscan and Dominican orders (1206 and 1216 respectively) who were at the vanguard of the fight against usury. The Church's attack on the charging of interest was in fact an attack on just the operations described and analyzed in Leonardo's book.[41]

Present Value Analysis

The most sophisticated of Fibonacci's interest rate problems is "On a Soldier Receiving Three Hundred Bezants for his Fief." In it, a soldier is granted an annuity by the king of 300 bezants per year, paid in quarterly installments of 75 bezants. The king alters the payment schedule to an annual year-end payment of 300. The soldier is able to earn 2 bezants on one hundred per month (over each quarter) on his investment. How much is his effective compensation after the terms of the annuity have changed? To solve this, Fibonacci explains

> First indeed you strive to reduce this problem to the method of trips and it is reduced thus… because there are 4 payments, 4 trips are similarly carried, and because each payment is 75 bezants, this is had for the expense of each

[40] Munro, John H., 2003, 'The Medieval Origins of the Financial Revolution: Usury, Rentes, and Negotiability', The International History Review, 25(3) September, pp. 505-62.

[41] There is some evidence in *Liber Abaci* that Fibonacci's financial analysis may have roots in the ancient Near East. Consider this problem: "A certain man gave one denaro at interest so that in five years he must receive double the denari, and in another five he must have double two of the denari and thus forever from 5 to 5 years the capital and interest are doubled. It is sought how many denari from this one denaro he must have in 100 years." Compare this to the problem from the Old Babylonian tablet 8528 in the Berlin museum published by Otto Neugebauer (1935). It asks: "If I lent one mina of silver at the rate of 12 shekels (a shekel is equal to 1/60 of a mina) per year, and I received in repayment, one talent (60 minas) and 4 minas. How long did the money accumulate?" [Sigler, 2002, Op.cit. p. 437] The solution is found by capitalizing interest only when the outstanding principal doubles. At 20%, the principal doubles every five years. Thus, the answer is that it would take 30 years for the debt to grow to 64 minae. The doubling over five year intervals seems a striking parallel, particularly given the fact that an annual compound rate that yields that result is not a round number – it is something less than 15%.

trip. Next because 53 is made from 50, you put $\frac{50}{53}$ for times for the four payments thus $\frac{50}{53}\frac{50}{53}\frac{50}{53}\frac{50}{53}\ldots$[42]

The product of these ratios is used to discount a payment by four periods (or trips). Notice that Fibonacci's novel fraction expression allows each term to be successively discounted by one more period. The discounted annual value of the 300 bezants paid in the last period is 259 and change. As before, Fibonacci explains how to construct a multi-period discount factor from the product of the reciprocals of the periodic growth rate of an investment, using the model developed from mercantile trips in which a percentage profit is realized at each city. In this problem, he explicitly quantifies the difference in the value of two contracts due to the timing of the cash flows alone. As such, this particular example marks the discovery of one of the most important tools in the mathematics of Finance – an analysis explicitly ranking different cash flow streams based upon their present value.

Institutional and Commercial Context

The soldier's present value problem above testifies to the fact that the mathematics of time-value discounting was apparently important 1202 when *Liber Abaci* was written. However, over the next few centuries its significance to governmental and commercial transactions grew tremendously. With the growth of international trade -- and warfare – in the late Middle Ages, the need for long-distance monetary transfers increased. Chinese merchants had solved this problem centuries earlier with the use of *feichan* "flying money" remittance certificates issued by government institutions in

[42] Sigler (2002) Op.cit, p. 392.

provincial capitals. Arab merchants in Fibonacci's time used an instrument much like a modern personal check for paper remittance. Neither of these explicitly involved a discount for the time-value of money. Early European money transfers -- bills of exchange -- did. Any distant remittance involved a time interval due to travel, and compensation for the use of the money over that period is natural, so bills of exchange were typically discounted to compensate for forgone interest revenue – essentially as in the example of "The Soldier Receiving Three Hundred Bezants for his Fief." Thus, bills of exchange were also interest rate instruments. Ashtor (1983) published the earliest known bill of exchange which appears to date from 1220.[43] Sivéry (1983) identifies an Italian discount contract with an implicit interest rate of around 11% from the year 1252.[44] Both the early 13th century date and the Levantine origin of Medieval bills of exchange form an interesting parallel to the publication date and Eastern influences of *Liber Abaci*. Were the mathematical tools for present value imported to Italy along with bills of exchange and Arabic numerals? Or did the technological innovation of using bills of exchange as debt securities (called "dry-exchange" by Medieval practitioners), stimulate mathematical work on the time value of money? In all likelihood, it was a fertile interplay between commerce and mathematical advancement, and Leonardo of Pisa was at its geographical and intellectual center.

Another key innovation occurred in northern Italy around Fibonacci's lifetime – the creation of long-term government debt. Venice and Genoa both began to

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[43] Ashtor, Eliyahu, 1983, *Levant Trade in the Later Middle Ages, Princeton*: Princeton University Press. Cited in Kohn, Meir, 1999, "Bills of Exchange and the Money Market to 1600," working paper Dartmouth College. http://www.dartmouth.edu/~mkohn/99-04.pdf. The latter is an up-to-date and excellent survey of bills of exchange.

[44] Sivéry, Gérard, 1983, "Mouvements de capitaux et taux d'interêt en occident au XIIIe siècle," *Annales Economies Societés Civilizations.* 38, p.367. Quoted in Spufford, Op.cit.

issue forced loans to her wealthy citizens in the late 12th Century. In 1171, for example, Venice issued a mandatory 5% loan to all citizens of the Republic to finance the construction of a fleet to fight Byzantium.[45] Out of this and successive loans called *prestiti* a regular market for government debt developed in Venice by the mid-13th century. The consolidation of the Venetian debt by decree in 1262 institutionalized the funding of municipal debt, and gave birth to the now widespread modern practice of financing the government through a national debt.

The roots of this debt revolution can be found in the financial records of Pisa itself. Pisa in Leonardo's time regularly made long-term debt commitments to creditors, promising the perpetual flow of revenues from specific sources to amortize a debt. Favier (1971) reports just such a contract with a creditor of the commune in 1173.[46] Historian David Herlihy points out that the public finances of the commune became increasingly complex through the course of the 13th Century. Pisan governors – *podesta* -- began to raise money through capitalization of taxation and monopoly rights.[47] The economic decision to alienate a stream of future cash flows in return for a current lump-sum payment is essentially a present value consideration. There is no doubt that the mathematical techniques in *Liber Abaci* would have allowed the *podesta* to quantify these decisions in a manner not feasible before Fibonacci's work. In fact, the annuity settled on Fibonacci late in his life for service to the state suggests that he may have had a hand in advising on precisely these financial contracts.

[45] Pezzolo, Luciano, 2003, "Bonds and Government Debt in Italian City States: 1250-1650," in Goetzmann and Rouwenhorst, eds. *Origins of Value: Innovations in the History of Finance*, forthcoming, Oxford Press.

[46] Favier, Jean, 1971, *Finance and Fiscalité au Bas Moyen Âge* , Société D'Édition D'Enseignement Supérieur, Paris, p. 283.

[47] Herlihy, David, 1958, *Pisa in the Early Renaissance: a Study ofUurban Growth*, Yale University Press, New Haven. Chapter 6 describes these new contracts. Pezzolo (forthcoming) Op.cit. cites examples of similar sales of income rights in Genoa in 1152 and Venice in 1164.

The technology of present value in *Liber Abaci* would also have served the analysis of private commercial transactions. Consider, for example, the extraordinary futures contracts quoted in neighboring Pistoia from at least the beginning of the 13th Century. From 1201 to 1210, contracts for delivery of perpetual delivery of a *staio* of wheat sold for a median price of 2.80 *lira* – a period in which, according to Herlihy, the "spot" price of a *staio* sold for ¼ of a *lira*. Without a mathematical method for discounting future deliveries, how did market participants determine the present value of such contract?[48] How were speculators able to arbitrage the relative prices of farmland and futures contracts? Indeed, did the capacity to discount future values exist in Northern Italy before Fibonacci or did his mathematics make futures and complex fixed income contracts possible? Ultimately, It is difficult to disentangle the causal relationship between the commercial problems of the day and the mathematical methods for solving them. Almost certainly, as is true today, there was a dynamic relationship between the analytical framework and the market. Mathematicians then, as now, must have been motivated by problems in the world around them, and in turn, their solutions engendered new advances in financial engineering.

Did Finance Influence the Mathematics of Infinite Series' or Vice-Versa?

The famous Fibonacci series problem is posed late in Chapter 12 – not as part of a series of related problems, but included after a problem on finding numbers equal to the sum of their factors, and before an algebra problem about four men with money. Is there any connection between the famous series and the commercial problems in the chapter? Certainly Chapter 12 is full of examples of compound growth from period to

[48] Herlihy, David, 1967, *Medieval and Renassiance Pistoia*, Yale University Press, New Haven, p. 138-9.

period – in particular, proportional increases in capital. The arithmetic of constructing a sequence of future values establishes a basis for considering a number of other, curious, infinite series.' One particularly interesting example of this sort in Chapter 12 is about the problem of doubling grains of wheat on a chessboard and summing the result – starting with only two grains, and doubling them through a chessboard of 64 squares leads to a quantity that is so great that "The number of ships [required to carry it] is effectively infinite and uncountable…"[49] Fibonacci marvels at how a simple growth algorithm applied to the well-known and finite pattern of a chessboard leads to imagining a vast, nearly infinite magnitude. This example suggests that Fibonacci's study of infinite series' may actually have developed as an extension of his analyses of compounding rates of return in finance problems.

After Liber Abaci

No matter how revolutionary, Leonardo's book did not suddenly advance Pisa to the economic forefront of competing Tuscan communes in the 13th Century. Although Pisa grew significantly during this period, she gradually lost out to other cites in the competition for the Levantine trade. Pisan merchants increasingly turned toward the North African colonies and trade with France and Spain, leaving trade with Constantinople and the Holy Land to her rivals Venice and Genoa. Larger political forces trumped any commercial or technological advantage potentially enjoyed as a result of Leonardo's writing and teaching.

One enduring aspect of Fibonacci's legacy is business education. Van Egmond (1980) documents a dramatic increase in the number of commercial arithmetic books in

[49] Sigler (2002), Op.cit. p. 437.

Italy during the centuries following the writing of *Liber Abaci* and Natalie Zemon Davis (1960) does the same for France.[50] Hundreds of hand-written and printed arithmetic books have survived from the European Renaissance, and many of them draw directly or indirectly from Leonardo's *Liber Abaci*. Undoubtedly beginning with Leonardo's own efforts to educate the citizens of Pisa in the 13th Century, Italian businessmen learned techniques of valuation and discounting according to his methods. When 15th century Florentine bankers extended their lending operations throughout Europe, they already had a centuries-long Italian tradition of financial mathematics upon which to draw.[51] To see how the mathematics of Finance developed after Fibonacci, let us consider two of the most important examples of his legacy of arithmetic writing and education: one from the 15th Century and another from the 16th Century.

The earliest extant printed commercial arithmetic is the *Treviso Arithmetic* of 1478, a volume of commercial problems written in the Venetian dialect, apparently used widely as a text for Renaissance Italian "Reckoning" schools. [52] *Liber Abaci* and the *Treviso* are clearly written in the same style – with an introduction to arithmetic operations, followed by chapters devoted to such topics as partnerships and division of profits, applications of ratios to trade and exchange, and metallurgy and coinage. Lacking in the *Treviso*, however are interest-rate problems and methods of discount. – perhaps reflecting the influence of usury laws in the preceding centuries curtailed their

[50]Van Egmond, Warren, 1980, *Practical Mathematics in the Italian Renaisance: A Catalogue of Italian Abbacus Manuscripts and Printed Books to* 1600, Monographia n. 4, Instituto di Storia Della Scienza Firenzi. Also, Davis, Nathalie Zemon, 1960, "Sixteenth-Century French Arithmetics on the Business Life." *Journal of the History of Ideas* (January-March), 21(1), pp. 18-48.
[51] For the evolution of arithmetic schools in the tradition of Fibonacci, see for example Poitras (2000) Op.cit and Swetz, Frank J., 1987, *Capitalism and Arithmetic; the New Math of the 15th Century*, Open Court, LaSalle.
[52] Swetz, Frank J. (1987) Op.cit. p. 293.

explicit role in mathematical education.[53] We know Italians were using discounting technology widely at this time, but apparently it would not do to include such examples in a published book.

By contrast, a book published a century later in Lyons draws heavily on the financial technology of *Liber Abaci*, and in fact demonstrates the fundamental role in European finance played by present value analysis. *L' Arithmetique* written by Jean Trenchant in 1558 follows the general structure of earlier and contemporary arithmetics. However, unlike the the author of the *Treviso*, Trenchant also devotes a major section of his book to interest rate and present value problems. Trenchant poses one problem in particular that clearly builds on Fibonacci's work, but is an actual "case" rather than simply an illustrative example.

> In the year 1555, King Henry, to conduct the war, took money from bankers at the rate of 4% per fair. That is better terms for them than 16% per year. In this same year before the fair of Toussaints, he received by the hands of certain bankers the sum of 3,945,941 ecus and more, which they called "Le Grand Party" on the condition that he will pay interest at 5% per fair for 41 fairs after which he will be finished. Which of these conditions is better for the bankers?[54]

The question is posed whether Henry's perpetuity at 4% per quarter is worth more or less than his 5% per quarter annuity for 41 quarters. Here, three and a half centuries after Leonardo, mathematicians are using methods developed in *Liber Abaci* to finance the struggle for control of Europe. Trenchant, in fact, may have been one of the

[53] See Poitras, 2000, Op. cit. p.160 for a discussion of the *Treviso*'s limited coverage of interest rate problems.
[54] Trenchant, Jean, *L'Arithmetique*, 1558, second edition, 1637, Lyons, p. 307. Author's translation.

mathematicians consulted in the structuring and valuation of these loans – he lived in Lyons where the financing took place.

Trenchant's solution is an elegant one. Rather than calculating the present value of the annuity and the perpetuity separately and comparing the two, he takes the incremental cash flow generated by the annuity over the 41 quarters and calculates the future value of that steam, under the assumption that each extra 1% payment per quarter is loaned at interest until the end of the period. He observes that, if the future value of this cumulated residual is less than the value of a perpetuity 41 quarters hence, then the 4% perpetuity is worth more than the annuity.

Earlier in his book he develops methods for calculating geometric progressions. He uses these methods to calculate a sequence of 41 numbers representing the future value of the 41 incremental cash flows due from the annuity, and then sums them and scales the value to the initial investment, allowing a comparison in percentage terms to any given initial investment in the perpetuity. He finds that the annuity is only worth 99.2% of the perpetuity, and thus concludes that the first deal is a better one for the bankers than the second.

Trenchant's analysis is noteworthy in that it demonstrates how important present value analysis had become in Europe by the 16th Century. Over the centuries following Leonardo, the present value method evolved from being the tool of an Italian commune seeking to reduce its pension expenses, to a critical factor in the financing of the defense of France. Unlike Leonardo who relied upon the analogy to trips, but who made a major contribution to the mathematics of infinite series', Trenchant's exposition of present value analysis was based directly upon solutions to geometric series. In fact, Trenchant is interesting is his own right. His development of present value analysis is more complete than that of Fibonacci – considering as it does the difference between an annuity and a perpetuity. Among other things, his book was the first to publish a table of discount factors for present value calculation.

East and West Revisited

Returning to the broader theme of the contrast between Europe and the Far East during the late Middle Ages and the Renaissance, it is curious that the interplay between mathematics and financial contracts witnessed in Europe from 1200 to 1700 was not matched in China, despite the advancement of both the Chinese financial system and Chinese mathematics – and indeed despite the cultural ties that linked China, India and the Middle East fostered by religious and commercial intercourse. Although the fragmentation of European monetary systems, the complexity of Mediterranean trade and the proscription again usury may also have been stimulating factors in the joint development of increasingly sophisticated financial tools and analytical methods, the largest institutional factor differentiating Europe and China in this period is public finance. James Tracy (1985) shows that, beginning around the turn of the 12th Century, cities like Venice, Genoa, Sienna and Florence in the South, and Douai, Valias and Ghent in Northern Europe all began to rely increasingly upon a variety of forms of debt financing.[55] In the North, cities sold long-term life-rent contracts to citizens, both perpetual and fixed term, creating a *rentier* class in the Late Middle Ages. In the South,

[55] Tracy, James, 1985, *A Financial Revolution in the Hapsburg Netherlands*, University of California Press, Berkeley, pp. 11-14.

cities issued callable bonds, non-callable bonds -- even bonds callable by lottery.[56] The assignability of these contracts led to active public securities trading – most spectacularly in 13th Century Venice near the Rialto bridge, where the first bond market was born.[57]

Even in the Middle Ages, public finance and private commerce relied upon an analytical system of valuation, and Fibonacci's book is the earliest historical evidence of such a system in Europe. While earlier evidence of such calculation may not have survived, the internal textual evidence in *Liber Abaci*, in particular the endogenous development of discounting methods from the analogy to trips, suggests that his contribution to the financial mathematics of the time was original.

History proceeds by processes of broad cultural trends, salient political and military events and population movements, as well as by discoveries and contributions by individuals. In China, there is little question that wars during the Song era stimulated a desperate need for innovations in governmental finance. Indeed, these pressures led to the invention in China of paper money, and the "discovery" of inflation – but not, interestingly enough, to state borrowing and deficit finance. Is this because China had no previous tradition of *census* contracts, or was it because of the diverging technological development of Finance and Financial Economics in the West, a development that eventually culminated in the far-flung capitalist empires of the late 19th Century? Had Fibonacci been born in China, would the world have become a different place?

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[56] Epstein, Steve A., 1996, *Genoa and the Genoese: 958-1528*, University of North Carolina Press, Chapel Hill, p. 192.

[57] See, Mueller, Reinhold, C., 1997, *The Venetian Money Market, Banks, Panics and the Public Debt 1200-1500*, Johns Hopkins University Press, Baltimore.

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Figure 1

CURRENCIES OF LATE MEDIEVAL EUROPE