Rebels, Conformists, Contrarians and Momentum Traders: Who Got It Wrong?

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Abstract

We consider investing in a noisy market with incorrect beliefs about predictability. Two types of agents use subjective models to optimize their portfolios — “conformists” who happen to believe in the self-fulfilling market consensus and “rebels” who have wrong beliefs. We compare the agents’ dynamic trading and their empirically observable investment performance. An agent who believes in log-normality is always a contrarian trader, who buys more shares after the price goes down, and sells shares when the price goes up. In contrast, an agent who believes in price predictability acts as a momentum trader, who buys more shares after the price goes up, for a range of subjective market mis-pricings. We show that more incorrect beliefs about predictability can lead to higher expected returns. Moreover, rebels with incorrect beliefs can have higher expected return than conformists with the same risk-aversion. Finally, it is more dangerous to be a sophisticated rebel in a non-predictable world, than to be a simplistic rebel in a predictable world.
Neo-classical asset pricing theory assumes that agents know the true specification of the stochastic discount factor. In reality, agents are imperfectly informed and cannot know the true specification. In this paper we consider a noisy market where agents invest based on their fixed subjective beliefs. The agents use different models to optimize their portfolios and execute different trades. We compare two types of agents — “conformists” who happen to believe in the self-fulfilling market consensus and “rebels” who have alternative beliefs, e.g. market predictability. What will a rebel do? How well will a rebel do? Can we distinguish the rebel from the market conformist? Which way to be wrong about market predictability is worse? The answers are surprising: for example, we find that the sophisticated rebel in an unpredictable world is worse off than a simplistic rebel in a predictable market.

Merton (1980) shows that noise prevents common knowledge of the true specification of the stochastic process for the market returns, specifically its mean. We consider agents who optimize their portfolios based on subjective model specifications and do not update\textsuperscript{1} these specifications over finite periods of time. For example, we can envision a “good, disciplined” fund manager, who performs due diligence on the fund’s asset class, (acquires the complete public information set), then makes an experienced judgment about future prospects (generates the subjective specification of the mean, possibly including predictability), decides on the best portfolio policy (optimizes) and finally executes it in a disciplined way (does not update the specification over a finite period). Moreover, we consider market predictability in the true specification or the agent’s different subjective specification.

Suppose the econometrician can observe only terminal wealth. Is it empirically feasible to determine whether an agent’s has subjective model is correct? How does

\textsuperscript{1}We assume that while the agent can be learning continuously, he uses a fixed specification for a finite period of time. For example, a new specification is adopted only after enough new data has been collected over a period of time to reject the current model at statistically significant levels.
the answer change if dynamic trading is also observable? These simple questions have profound implications for the prevalence of market efficiency. This is so, because potential limitations on the learning process would affect the aggregation of imperfect information in the price, and empirical inference about belief correctness is a critical part of the learning process of market participants. Our main objective is to establish whether mis-specified beliefs can be detected empirically given a limited observational ability.

The second contribution of our paper is to derive quantitative limitations on conventional investment performance evaluation. Is it meaningful to use static performance measures like the Sharpe ratio when the prices and the optimal trading are continuous processes? We highlight one application of our results in the ”skill versus luck” debate about observed mutual funds performance.

Third, we ask which world is more forgiving to rebels. Is it more dangerous to be a ”naive” rebel who believes there is no predictability, while the world truly is predictable, as opposed to a sophisticated rebel who believes in predictability when the true world is not predictable? We are agnostic about the true market process and consider two scenarios: a log-normal market and second, a log-price that follows a trending Ornstein-Uhlenbeck process that has long-term predictability. We compare the two fallacies in terms of reward and risk. The fallacy of being “naive” is to use a log-normal specification in the predictable world. The opposite fallacy is to over-complicate the specification with predictability when the true world is log-normal.

Our first set of results shows that when it is feasible to observe the continuous trading of an agent who believes in log-normality, the agent is always a locally contrarian trader, who buys more shares after the price goes down, and sells shares when

\footnote{Our agnostic approach is consistent with our assumption that market is noisy, which is justified further by the empirical difficulties in estimating the mean of the market process, see e.g. Merton (1980), Summers (1986) and Poterba and Summers (1988).}
the price goes up. In contrast, an agent who believes in price predictability acts as a momentum trader for a range of prices that correspond to a range of perceived market mis-pricing. Intuitively, the agent’s perceived opportunity set is changing with the price level, so the agent is ”timing” the market and buys more shares after the price goes up in order to reach the optimal allocation for the new opportunity set.

Our results on the expected terminal wealth of the agents show that it is more dangerous to be a sophisticated rebel in a simple world, than to be a simplistic rebel in a predictable world. The over-specified rebel who believes in predictability when the world is log-normal has decreasing expected returns when either the over-pricing or the under-pricing is increasing. Intuitively, the rebel is wrongly timing the market when both the price appears to be too low and when it appears too high.

In contrast, the simplistic rebel, who believes in a constant mean of the market return in a predictable world with trend-reverting log-prices is not punished as severely, because his expected return decline is not symmetrical in the mis-pricing. The rebel’s expected return decreases when the market is over-priced, but it is rising when the market is under-priced.

Our paper is related to several modern developments in the revealed preferences literature, especially Leland (1996,1980), Dybvig and Rogers (1997) and Cuoco and Zapatero (2000). Our paper has one main distinction from this literature: we consider the empirical implications of beliefs about market predictability. Moreover, addressing this different set of issues requires a distinct analytical approach to handle path-dependent portfolios.

Our first motivation is closest to Leland (1996) who ”reverse-engineers” demand for options to impute different expectations to agents that demand different options. Our paper explores a different question: what are the implications of particular beliefs about predictability for the empirically observable investment performance? In particular, we focus on the case when continuous trading is not observable and hence
reverse-engineering of dynamic trading strategies is not empirically feasible. Instead, the empirically observable variables typically consist of wealth observed with finite frequency that usually spans periods of unobservable trading, e.g. the monthly performance numbers of a mutual fund that re-balances its portfolio daily. Accordingly, we consider the properties of the terminal wealth that is accumulated following an optimal trading strategy over a finite period of time\textsuperscript{3}. We compare different subjective beliefs in terms of the moments, i.e. "reward" and "risk," of the terminal wealth distribution\textsuperscript{4}.

Our approach can be viewed as a powerful generalization of the seminal Leland (1980) model of agents who optimize using biased estimates of the mean\textsuperscript{5}. Leland’s model assumes that agents know and use the correct model specification, which is the path-independent log-normal process\textsuperscript{6}. However, the long-standing difficulties in empirically estimating expected returns suggest that one cannot readily ascertain what is the true specification for the mean. Moreover, our main goal is to study the optimal trading and performance with different beliefs about predictability. Hence, we consider beliefs in a more general sense. We posit different beliefs about the process specification and not just the parameter values\textsuperscript{7}. In particular, we model

\textsuperscript{3}In contrast, Leland’s reverse-engineering of a given optimal demand for options produces the implied probabilities on a fixed binomial tree with path-independent transitional probabilities.

\textsuperscript{4}Wang (2002) uses a Bayesian setting to examine how an agent’s statistical estimates of the mean and variance are affected by the agent’s particular prior beliefs about the relative importance between a model and the data.

\textsuperscript{5}Leland established that when agents disagree about the parameter estimate of the mean, they trade on the opposite sides, because they have different demand for portfolio insurance. See also Brennan and Solanki (1981)

\textsuperscript{6}In effect, the agents have enough information to know the true model, but cannot compute unbiased estimators for the model parameters, i.e. the constant mean.

\textsuperscript{7}A case in point is Leland’s agents who both believe in the log-normal specification, but with different values of the mean parameter. The agents have the same beliefs in market non-predictability.
beliefs where the investment opportunity set is changing through time, i.e. the mean is not constant.

Our focus on beliefs about market predictability dictates the introduction of a major innovation: beliefs with path-dependent transitional probabilities. The modern literature is confined to the classical case of path-independent wealth. Moreover, Dybvig and Rogers (1997) and Cuoco and Zapatero (2000) study the feasibility of recovering preferences from observed optimal consumption policies, given a fixed set of beliefs. In contrast to these papers, we focus on the opposite problem of recovering mis-specified beliefs for fixed preferences, and also on the implications of a mis-specified optimization for investment performance.

The paper is organized as follows. In section I, we set up the model and derive the optimal portfolio policies for agents with different beliefs about the market process. Section II analyzes the local and global appearance of the optimal portfolio policies as momentum or contrarian trading. Section III examines the performance of the strategies that are optimal under different beliefs in two different worlds. Section IV compares rebels in different worlds and contains a qualitative discussion of the feasibility of empirically distinguishing between different beliefs. Section V concludes.

I. THE MODEL AND THE OPTIMAL POLICIES

A. Information Structure

We assume that there is a single risky asset, the market portfolio, with price $P$ at time $t$ and a riskless asset with a constant risk-free rate $r$. The usual continuous time

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8Dybvig and Rogers' (1997) empirically relevant results require log-normality and path-independence. Cuoco and Zapatero (2000) present a general proof of the uniqueness of recovered preferences, but do not cover the case of no intermediate consumption. See also He and Leland (1992) and Wang (1993a,b) on recovering the preferences of the representative agent.
primitives are denoted as follows. Let \( z \) be the standard one-dimensional Brownian motion on \( \Omega \times [0, T], T \leq \infty \), where \((\Omega, F, \mathcal{F})\) is a complete probability space with the standard filtration \( F = \{ F_t : t \in [0, T]\} \). The market price process is adapted to \( F \). When there is no ambiguity we will suppress the subscript for the current time \( t \).

Let \( p = \ln P \) follow the process

\[
dp = \mu dt + \sigma dz
\]

\[
\mu \equiv \gamma - \alpha(p - \gamma t)
\]

From Ito’s lemma, we find the drift

\[
d\mu = -\alpha(\mu - \gamma)dt - \alpha\sigma dz
\]

and also the price that exhibits long-term reversion to an exponential trend:

\[
\frac{dP}{P} = \left[ (\gamma + \frac{\sigma^2}{2}) - \alpha(\ln P - \gamma t) \right] dt + \sigma dz
\]

\[
\text{Cov}(d\mu, \frac{dP}{P}) = -\alpha\sigma^2
\]

We set

\[
X_t \equiv p(t) - \gamma t
\]

so that for \( 0 \leq t \leq T \) we have

\[
X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-u)} dz(u)
\]

\[
\sim N \left( X_0 e^{-\alpha t}, \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \right)
\]

Note that when \( \alpha \to 0 \), this specification reduces to the log-normal process:

\[
dp = \gamma dt + \sigma dz
\]

with constant drift \( \gamma \).


B. Beliefs About Predictability

Agent are not endowed with perfect knowledge of the true process specification. Each agent knows that any agent’s information is incomplete and of unknown precision, hence there are no strategic games. Agents invest based on their privately generated subjective beliefs. There are two types of agents, who cannot affect the price individually. The market conformists have the correct beliefs. The rebels have incorrect beliefs and do not adapt to the correct beliefs. We assume that all agents agree on the value of the diffusion parameter $\sigma$ and can differ only in their specification of the drift. In particular, an agent believes that prices are predictable if $\alpha = 0$. Conversely, log-normal beliefs mean $\alpha = 0$.

C. The Optimum Portfolio

For a power utility of terminal wealth at time $T$ and no intermediate consumption,

$$U(W_T) = \frac{1}{1 - c^2} W_T^{1-c^2}, c > 1$$

the derived utility $J$ is partly separable

$$J(W, \mu, t) = Q(\mu, t) \frac{W_T^{1-c^2}}{1 - c^2}$$

Solving the simplified Bellman equation, we obtain the optimal portfolio, (see the appendix or Merton (1971)),

$$w = -\frac{J_W}{WJ_{WW}} \mu + \frac{\sigma^2}{2} - r + \frac{J_{\mu W}}{WJ_{WW}} \alpha$$

$$= \frac{\mu + \frac{\sigma^2}{2} - r}{c^2 \sigma^2} + H$$
where the first term is the optimal policy for a log-normal process and the second term is the hedging demand $H$ for the predictability:

$$H \equiv -\frac{\alpha Q\mu}{c^2 Q},$$

We have (see the appendix),

$$H = -(c-1)(1-e^{-\frac{\mu}{\sigma}(T-t)}) \times (c+1) \frac{1 - \frac{c-1}{c+1}e^{-2\frac{\alpha}{\sigma}(T-t)}}{1 + \frac{c-1}{c+1}e^{-2\frac{\alpha}{\sigma}(T-t)}} \frac{\mu + \frac{\sigma^2}{c^2} - r}{\frac{1}{c^2\sigma^2} + \frac{c-1}{c+1}e^{-2\frac{\alpha}{\sigma}(T-t)}} - \frac{1 + e^{-\frac{\mu}{\sigma}(T-t)} \alpha X_t}{1 + \frac{c-1}{c+1}e^{-2\frac{\alpha}{\sigma}(T-t)} c^2\sigma^2}$$

The optimal portfolios for the agents are as follows.

**Proposition 1.** The path-dependent optimal portfolio is

$$w(t) = a(t) - b(t)X_t$$

$$a(t) = \bar{w} + \bar{w}(c^2 - 1)(1 - e^{-\frac{\mu}{\sigma}(T-t)}) \frac{1 - \frac{c-1}{c+1}e^{-\frac{\alpha}{\sigma}(T-t)}}{1 + \frac{c-1}{c+1}e^{-2\frac{\alpha}{\sigma}(T-t)}}$$

$$b(t) = \frac{\alpha}{c^2\sigma^2} \frac{1 - \frac{c-1}{c+1}e^{-\frac{\alpha}{\sigma}(T-t)}}{1 + \frac{c-1}{c+1}e^{-2\frac{\alpha}{\sigma}(T-t)}}$$

$$\bar{w} \equiv \frac{\gamma + \frac{\sigma^2}{2} - r}{c^2\sigma^2}$$

and $a \to \bar{w}, b \to 0$ as $\alpha \to 0$. As a corollary, if $\alpha = 0$ then the price is log-normal and the optimal policy $w$ reduces to the constant fraction policy $\bar{w}$.

**Proof:** see the appendix.9

9The solution can also be obtained as a special case of Kim and Omberg (1996). See also Wachter (2002).
Proposition 2. The optimal portfolio for a log-normal specification is

\[ w_{\alpha=0}(t) = \bar{w} \]

Note that there is no hedging demand for the log-normal specification: \( H = 0 \).

The two different policies are visualized on Figure 1.

Figure 1

The figure shows that larger market exposures correspond to more extreme beliefs about faster reversion to the long-term trend. We now consider the observed trading of the agents.

II. MOMENTUM TRADING

We have the following definition of momentum trading..

A. Definitions

Let \( n \) be the number of shares priced at \( P \),

\[ n = \frac{wW}{P} \]

Using stochastic differentials we have

Definition I.

Momentum Trade:

\[ dP > 0 \text{ implies } dn > 0 \]
\[ dP < 0 \text{ implies } dn < 0 \]

Contrarian Trade:

\[ dP > 0 \text{ implies } dn < 0 \]
\[ dP < 0 \text{ implies } dn > 0 \]
Denote the surprises in the return $\frac{dP}{P}$ and in the percentage change $\frac{dn}{n}$ in the number of shares as

$$\frac{\tilde{d}P}{P} \equiv \frac{dP}{P} - E\left[\frac{dP}{P}\right] = \sigma dz$$

$$\frac{\tilde{d}n}{n} \equiv \frac{dn}{n} - E\left[\frac{dn}{n}\right] = \sigma_N dz_N$$

Now, since locally the realized return is a.s. of the order of the unexpected return $d\tilde{P}/P$

$$\frac{\tilde{d}P}{E[\tilde{d}n]} = \frac{\sigma dz}{\frac{\sigma \varepsilon \sqrt{dt}}{\mu dt}} = \frac{\sigma}{\mu} \varepsilon \sqrt{dt}, \varepsilon \sim N(0,1)$$

and similarly $dn/n$ is a.s. of the order of $d\tilde{n}/n$ we have the following equivalent definitions:

**Definition II.**

**Momentum Trade:**

$$\frac{\tilde{d}P}{P} > 0 \text{ implies } \frac{\tilde{d}n}{n} > 0$$

$$\frac{\tilde{d}P}{P} < 0 \text{ implies } \frac{\tilde{d}n}{n} < 0$$

**Contrarian Trade:**

$$\frac{\tilde{d}P}{P} > 0 \text{ implies } \frac{\tilde{d}n}{n} < 0$$

$$\frac{\tilde{d}P}{P} < 0 \text{ implies } \frac{\tilde{d}n}{n} > 0$$

Next, set

$$F(P,t) \equiv \frac{w(P,t)}{P}$$

and using $n = FW$ and $F_p = (\frac{dn}{dP}P - w)/P^2$, we have
\[ dF = F_P dP + \left( F_t + \frac{1}{2} F_{PP} \sigma^2 P^2 \right) dt \]
\[ = \left( F_P \mu P + F_t + \frac{1}{2} F_{PP} \sigma^2 P^2 \right) dt + \left( \frac{\frac{dP}{dP} - w}{P^2} \right) \sigma P \, dw \]

then, using this and \( dW = wW dP/P + (1 - w)Wr dt \) with Ito's lemma, we get

\[ dn = W dF + F dW + \sigma f \sigma w W dt \]
\[ = (\cdot) dt + \sigma w W \left[ \frac{1}{w} \frac{dw}{dP} P + w - 1 \right] dz \]

which can be written as

\[ \frac{dn}{n} = (\cdot) dt + \left[ \frac{1}{w} \frac{dw}{dP} P + w - 1 \right] \sigma dz \]

or finally,

\[ \tilde{dn} = \left[ \frac{1}{w} \frac{dw}{dP} P + w - 1 \right] \frac{dP}{P} \]

and we have proved

**Theorem 1:** Momentum trading is equivalent to the condition

\[ \delta \equiv \frac{1}{w} \frac{dw}{dP} P + w - 1 > 0 \]

We now consider the trading policies in section I.
B. Momentum Trading Results

Consider the two different portfolio policies from section I. First, an agent with log-normal beliefs optimally invests a constant fraction of wealth in the market. In the other case, the subjective mean-reverting specification implies a path-dependent optimal portfolio policy.

Case 1. Constant portfolio weight, \( w = \bar{w} \) (e.g. optimal portfolio for log-normal beliefs).—

If the fraction of wealth \( w \) invested in the market is constant at all times, then

\[
\delta_{\bar{w}} = 1 \frac{d\bar{w}}{d\bar{P}} + \bar{w} - 1 = \bar{w} - 1
\]

so that as long as the agent is sufficiently risk-averse not to be levered,

\[
\bar{w} < 1
\]

we have

\[
\frac{\tilde{d}n}{n} = -(1 - \bar{w}) \frac{\tilde{d}P}{P}
\]

so that we have proved

Proposition 3. An agent who invests a constant fraction of wealth in the market is always locally a contrarian trader, regardless of the particular specification for the mean of the return process.
Case 2. Portfolio weight $w$ is path-dependent and linear in the current mis-pricing, (e.g. portfolio that is optimal for beliefs in predictability).—

$$w(t) = A(t) - B(t)X_t, B(t) > 0$$

$$X_t \equiv p(t) - \gamma t$$

e.g. $A(t) = a(t), B(t) = b(t)$ in the optimal portfolio for beliefs in predictability that was considered above. We have,

$$\frac{dw}{dP} = \frac{B}{P}$$

so that

$$\delta \equiv \frac{1}{w^2} \frac{dw}{dP} P + w - 1 = \frac{1}{w^2} w(w - w_+)(w - w_-)$$

$$w_- \equiv \frac{1 - M}{2} < 0$$

$$w_+ \equiv \frac{1 + M}{2} > 0$$

$$M^2 \equiv 1 + 4B > 1,$$

Since the numerator of $\delta$ is a cubic expression, $\delta$ changes sign twice and there are four signed regions. From Theorem 1, the agent executes momentum trades if

$$w_- < w < 0$$

$$0 < w_+ < w$$

and contrarian trading otherwise.

The conditions for momentum can also be expressed as a condition on the (perceived) mis-pricing $X_t \equiv \ln P - \gamma t$. The momentum trading condition above is equivalent to

$$w - w_- = A - BX_t - \frac{1 - M}{2} = -B \left( X_t - \frac{A}{B} - \frac{\sqrt{1 + 4B} - 1}{2B} \right) > 0$$
and

\[ w - w_+ = A - BX_t - \frac{1 + M}{2} = -B \left( X_t - \frac{A}{B} + \frac{\sqrt{1 + 4B + 1}}{2B} \right) > 0 \]

so that using these results, we obtain

**Proposition 4.** An agent who invests a path-dependent fraction \( w = A - B(p - \gamma t) \) of wealth in the market executes *momentum trades* when

\[ \frac{A}{B} < X_t < \frac{A}{B} + \frac{\sqrt{1 + 4B - 1}}{2B} \]

Note that for a fixed mean-reversion parameter \( \alpha \), the path-dependent policy is momentum trading for a range of perceived mis-pricing parameters \( X_t \). In contrast, the fixed fraction policy is always locally contrarian.

The above result is not an "exotic" scenario that requires extreme parameter values. As an example, consider an agent whose utility exponent is -3 (so that \( c = 2 \)), with the (annual, \( T = 1 \)) riskless rate \( r = 0.05 \) and suppose that the long-term growth trend \( \gamma = 0.15 \), the diffusion parameter \( \sigma = 0.25 \) and the under-pricing is expected to be corrected half-way in about 4 months, so the reversion speed is \( \alpha = 2 \). Momentum trading is optimal if the agent believes that the mis-pricing is either \( X < 0.2049 \), corresponding to a subjective under-pricing of at least 18.53%, or \( 0.0931 < X < 0.3227 \), corresponding to a subjective over-pricing between 9.76% and 38.08%. This completes the characterization of locally-observable trading behavior.
III. TERMINAL WEALTH

We set, as before,

\[ X_t \equiv p(t) - \gamma t \]

so that by Ito’s lemma, we have

\[ dX = -\alpha X dt + \sigma dz \]

and the optimal portfolio policy for this specification is linear in \( X \):

\[ w(t) = a(t) - b(t)X_t \]

Moreover, if \( \alpha = 0 \) the price is log-normal and \( a = \bar{w}, b = 0 \), where as before we denote \( \bar{w} \equiv (\gamma + \frac{\sigma^2}{2} - r)/c^2\sigma^2 \).

The standard dynamic equations can be expressed in terms of \( X \). We have

\[
\frac{dP}{P} = (\gamma + \frac{\sigma^2}{2} - \alpha X)dt + \sigma dz
\]

\[
\frac{dW}{W} = \left( r + [(\gamma + \frac{\sigma^2}{2} - r) - \alpha X]w \right) dt + w\sigma dz
\]

and using \( \gamma + \frac{\sigma^2}{2} - r = c^2\sigma^2\bar{w} \), we can apply Ito’s lemma for \( Y = \ln W \) to get

\[ dY = [r + (c^2\sigma^2\bar{w} - \alpha X)w - \frac{\sigma^2}{2}w^2]dt + w\sigma dz \]

In particular, the continuously-compounded internal rate of return (the return) is

\[
R_T \equiv \ln \left( \frac{W_T}{W_0} \right) = \frac{Y_T - Y_0}{Y_0} = rT + \int_0^T [(c^2\sigma^2\bar{w} - \alpha X)w - \frac{\sigma^2}{2}w^2]dt + \sigma \int_0^T wdz
\]

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We will focus on the risk-reward trade-off between the expected value and the variance of $R_T$.

**A. The Log-normal World**

We have, e.g. if we set $\alpha = 0$ in the mean-reverting specification, for $0 \leq t \leq T$

$$dX = \sigma dz$$

$$X_t \sim N \left( X_0, \sigma^2 t \right)$$

so that the return is

$$R_T = rT + \sigma^2 \int_0^T (c^2 \bar{w}w - \frac{1}{2}w^2) dt + \sigma \int_0^T wdz$$

First, we consider the log-normal conformist, followed by the trend-reverting rebel.

**A.1. The Conformist.**—

The log-normal conformist optimally invests a constant fraction of wealth in the market:

$$w = \bar{w}$$

and the internal rate of return for the conformist is

$$R_T = [r + (c^2 - \frac{1}{2})\bar{w}^2 \sigma^2]T + \bar{w} \sigma \int_0^T dz$$

Moreover, $R_T$ is normally distributed and its expected value and variance are

$$E[R_T] = [r + (c^2 - \frac{1}{2})\bar{w}^2 \sigma^2]T$$

and

$$Var[R_T] = \bar{w}^2 \sigma^2 T$$
The conformist Sharpe ratio, which measures the “reward-to-risk” tradeoff is

\[
\frac{E[R_T] - rT}{\sigma[R_T]} = (1 - \frac{1}{2c^2})\gamma + \frac{\sigma^2}{\sigma} - r\sqrt{T}
\]

Note that for the conformist, higher risk aversion implies higher Sharpe ratio. The Sharpe ratio is always lower than the limit Sharpe ratio for infinite risk aversion, \(c \to \infty\), which is \(\frac{\gamma + \sigma^2/\sigma}{\sigma} - r\sqrt{T}\).

In this case, the terminal wealth is path-independent and hence is available explicitly:

\[
W_T = W_0 e^{(1-\bar{w})(r + \bar{w}\frac{\sigma^2}{2})T} \left( \frac{P_T}{P_0} \right)^{\bar{w}}
\]

We can use the conformist as a benchmark for the rebel.

A.2. The Rebel.—

The rebel’s terminal wealth is path-dependent due to the path-dependent optimal portfolio \(w(t), 0 \leq t \leq T\) (see Proposition 1)

\[w(t) = a(t) - b(t)X_t\]

Suppressing the time notation in the coefficients \(a, b\), we write the internal rate of return is

\[
R_T = rT + \sigma^2 \int_0^T [c^2 \bar{w} (a - bX_t) - \frac{1}{2} (a - bX_t)^2]dt + \sigma \int_0^T (a - bX_t)dz
\]

\[
= rT + \sigma^2 \int_0^T \left[ -\frac{1}{2} b^2 X_t^2 + (a - c^2 \bar{w})bX_t + (c^2 \bar{w} - \frac{1}{2} a) \right]dt + \sigma \int_0^T (a - bX_t)dz
\]

We have
Proposition 5. The expected return of the rebel is

\[ E[R_T] = J(\alpha)X_0^2 + K(\alpha)X_0 + L(\alpha) \]

where

\[ J(\alpha) = -\frac{T}{2c^2\sigma^2}\alpha^2 \left(1 - \frac{1 - e^{-\frac{2\alpha T}{c}}}{\frac{2\alpha T}{c}} \frac{\frac{2c-1}{c+1}}{e^{-\frac{2\alpha T}{c}}} \right) < 0 \]

\[ K(\alpha) = -\bar{w}(c^2 - 1)(1 - e^{-\frac{\alpha T}{c}}) \frac{1 - \frac{c-1}{c+1}e^{-\frac{\alpha T}{c}}}{1 + \frac{c-1}{c+1}e^{-\frac{2\alpha T}{c}}} < 0 \]

\[ L(\alpha) = -\frac{T^2}{4c^2}\alpha^2 + \frac{c-1}{2c^2}T\alpha + \left( r + \frac{1}{2}c^4\sigma^2\bar{w}^2 \right)T - \]

\[ \frac{c\sigma^2\bar{w}^2T \left(1 - e^{-\frac{\alpha T}{c}}\right)(c^2 - 1)(c - 1)}{\frac{2\alpha T}{c}} \frac{1 - \frac{c-1}{c+1}e^{-\frac{2\alpha T}{c}}}{1 + \frac{c-1}{c+1}e^{-\frac{2\alpha T}{c}}} - \frac{1}{2} \ln \left( \frac{1 + \frac{c-1}{c+1}e^{-\frac{\alpha T}{c}}}{1 + \frac{c-1}{c+1}e^{-\frac{2\alpha T}{c}}} \right) \]

Proof: in the Appendix.

First, note that the rebel’s expected return \( E[R_T] \) is a quadratic function of the subjectively (and wrongly) estimated initial mis-pricing \( X_0 \). Thus, larger absolute mis-pricing \( X_0 \) has the same effect on the expected return, regardless of whether the market is over-priced (\( X_0 > 0 \)) or under-priced (\( X_0 < 0 \)). The effect depends only on the sign of the quadratic coefficient \( J(\alpha) \). Since \( J \) is negative, larger subjective mis-pricing implies lower expected returns and vice versa. Note that the negative sign of the coefficient \( J(\alpha) \) does not depend on the agent’s beliefs through the speed \( \alpha \) and it does not depend on the risk aversion parameter \( c \).

Next, observe that since in general \( K(\alpha) \neq 0 \), the rebel’s highest expected return is not obtained if the rebel happens to believe that there is no mis-pricing, \( X_0 = 0 \). Intuitively, the rebel believes in reversion, hence belief in subjective under-pricing produces higher returns. We return to this intuition later.

Proposition 6. The variance of the rebel is

\[ Var[R_T] = M(\alpha)X_0^2 + N(\alpha)X_0 + Q(\alpha) \]

\( M(\alpha) > 0 \)
Proof: in the Appendix.

Note that the rebel’s total risk, \( V_{ar}[R_T] \), also is a quadratic function of the initial mis-pricing \( X_0 \). Thus, larger absolute subjective mis-pricing \( X_0 \) has the same effect on risk, regardless of the sign of the mis-pricing. The effect depends only on the sign of the quadratic coefficient \( M(\alpha) \). Since \( M \) is negative, larger subjective mis-pricing implies higher total risk and vice versa. Again, the positive sign of the coefficient \( M(\alpha) \) does not depend on the agent’s beliefs through the speed \( \alpha \) and it does not depend on the risk aversion parameter \( c \).

We can write the rebel’s Sharpe ratio as

\[
\frac{E[R_T] - rT}{\sigma[R_T]} = \frac{J(\alpha)|X_0| - K(\alpha) \text{sign}(X_0) - L(\alpha)/|X_0|}{\sqrt{M(\alpha) + N(\alpha)/X_0 + Q(\alpha)/X_0^2}}
\]

and since \( J < 0 \), we have

\[
\frac{E[R_T] - rT}{\sigma[R_T]} \rightarrow -\infty \text{ as } X_0 \rightarrow \pm \infty
\]

Observe that the risk-reward tradeoff deteriorates without limit as the perceived mis-pricing increases.

A.3. Performance Comparison.—

In the log-normal world, the rebel by definition believes in predictability. The trend-reverting predictability that we consider has two aspects: first, the subjective mis-pricing, measured by the initial deviation from the trend, \( X_0 \), and second, the rate at which the log-price reverts to the trend, measured by the speed parameter \( \alpha \). To quantify the effects of wrong beliefs, as reflected in the model mis-specification,
we can fix the reversion speed (e.g. $\alpha = 1$) over a fixed time period $T = 1$ (e.g. one year) and examine the comparative statics of the expected return, the variance and their ratio with respect to the mis-pricing $X_0$. Figure 2 shows the rebel’s expected return $E[R_T]$ as a function of the subjective mis-pricing $X_0$ for three different values of $\alpha$.

Figure 2

The figure illustrates the following comparative static result.

Proposition 7. There is a range of values for the subjective under-pricing $X_0$, where the rebel’s expected return is *increasing* with the under-pricing. The rebel’s expected return is always decreasing when the absolute value of the mis-pricing, $|X_0|$ is increasing beyond a fixed threshold value $X_0^*$.

Proof: We show in the appendix that since the sign of the quadratic coefficient $J$ is negative and $K < 0$, the maximum is obtained at a value $X_0^{\text{max}} \equiv -K/2J$ that is strictly less than zero. Hence, the expected return is decreasing on the segment $[X_0^{\text{max}}, 0]$ of under-pricing values, i.e. it is increasing as the (negative) under-pricing is increasing.

Proposition 7 reveals a surprising finding: in some cases, more incorrect beliefs lead to higher expected return! For a non-empty range of under-pricing values, the expected wealth *increases* as beliefs become more incorrect! Moreover, this phenomenon *always* occurs for *any* degree of risk-aversion!

Also, note that the result is not a remote possibility in the sense that it holds for ”reasonable” mis-pricing parameter values. As an example, consider an agent whose utility exponent is -3 (so that $c = 2$), with the (annual, $T = 1$) riskless rate $r = 0.05$ and suppose that the current price $P_0 = 45$, with a long-term growth trend $\gamma = 0.15$ and instantaneous volatility $\sigma = 0.25$. The agent believes that the price should be priced at 50, so that $X = \ln(45/50) = -0.1054$ and the under-pricing will
be corrected half-way in about 4 months, so the reversion speed is \( \alpha = 2 \). In this case, the expected return is \( E[R_T] = 0.0593 \). On the other hand, if the agent believed that the price was not off the long-term trend, i.e. \( X = 0 \), the expected return would be lower: \( E[R_T] = 0.0233 \).

This result has grave implications for performance evaluation: a sample of high realized returns may reflect investment with wrong beliefs about predictability. In contrast, higher realized returns (past performance) are conventionally interpreted as evidence of better operational efficiency i.e. judgement and/or information, for example in the case of mutual fund performance evaluation.

We now consider whether the rebel have an expected return that is greater than the expected return of a conformist with the same risk-aversion. We have

**Proposition 8.** The expected return of the rebel can be higher than the constant expected return of a conformist with the same risk aversion.

Proof: Since the rebel’s expected return, \( E[R_T] = JX_0^2 + KX_0 + L \) is a quadratic function of \( X_0 \) with a negative quadratic coefficient, \( J < 0 \), it suffices to show that the maximum of the quadratic function, \( (-K^2/4J) + L \), can be greater than the conformist’s expected return. We demonstrated in the appendix that as \( \alpha \to 0 \), the difference between the rebel’s and the conformist’s expected return tends to a positive limit of \( \frac{1}{2}(c^2 - 1)^2\sigma^2w^2T \).

Proposition 8 reveals another surprising finding: rebels with incorrect beliefs can have higher expected return than conformists with the same risk-aversion! Moreover, the performance lead of the rebel is higher for higher risk aversion! And once again, this phenomenon always occurs for any degree of risk aversion, for sufficiently moderately wrong beliefs!

As an example, consider a utility function with \( c = 2 \), along with the parameter values from the previous example. If the reversion speed is \( \alpha = 1 \), corresponding to an expected half-way correction of the under-pricing in about 8.5 months, then the
expected return of the rebel, $E[R_T^{reb}] = 0.1428$, is higher than the expected return of
the conformist, $E[R_T^{con}] = 0.1103$.

These results justify further caution with performance evaluation: one manager’s
high realized returns need not imply that he was ”right” and his lower-performing peer
was wrong – we have just showed that in fact, superior results may reflect investment
with more incorrect beliefs about predictability. In contrast, higher realized returns
(past performance) are conventionally interpreted as evidence of superior skill relative
to peers. The apparent intuition of this result is that the rebel who invests with more
incorrect beliefs assumes more risk and is efficiently rewarded with higher expected
return. However this intuition is mis-leading, since increasing the incorrectness of
beliefs even further leads to lower and eventually negative expected returns. The key
observation is that a particular belief about the underpricing does not merely imply
higher risk-taking, but it affects the entire subsequent dynamic trading throughout
the investment horizon. This also confirms the classical paradigm that the proper
measure of performance cannot be returns alone, but needs to involve risk and the
trade-off between the two.

Next we compare total risk measured by the variance of terminal wealth.

Proposition 9. The rebel’s total risk is increasing with the perceived mis-pricing.

Proof: The rebel’s variance of return, $Var[R_T] = MX_0^2 + NX_0 + Q$ is a quadratic
function of $X_0$ with a positive quadratic coefficient, $M > 0$.

This result supports the intuition that rebels with more extreme beliefs run more
significant risk exposures. The market is efficient, in the sense that extremist rebels
are not likely to survive. We now examine the effect of beliefs about predictability
on the risk-reward trade-off.
Proposition 10. The rebel’s reward-to-risk trade-off is decreasing with increasing subjective mis-pricing. The rebel’s risk reward-to-risk trade-off is always worse than the conformist, who has a constant reward-to-risk trade-off.

Proof: The first part of the proposition was demonstrated above, the second part is demonstrated in the appendix.

The rebel who believes in more extreme mis-pricing suffers both a lower expected return and a higher variance.

B. The Trend-Reverting World

We have $\alpha > 0$ in the mean-reverting specification

$$dX = -\alpha X dt + \sigma dz$$

and for $0 \leq t \leq T$

$$X_t \sim N \left( X_0 e^{-\alpha t}, \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t}) \right)$$

so that the return is

$$R_T = rT + \int_0^T [(c^2 \sigma^2 \bar{w} - \alpha X_t)w - \frac{\sigma^2}{2}w^2] dt + \sigma \int_0^T wdz$$

First, we consider the trend-reverting conformist, followed by the log-normal rebel.

B.1. The Conformist.—

We substitute the path-dependent optimal portfolio $w(t), 0 \leq t \leq T$, from Proposition 1,

$$w(t) = a(t) - b(t)X_t$$

Suppressing the time notation in the coefficients $a, b,$ the internal rate of return is
\[ R_T = rT + \int_0^T \left[ \left( c^2 \sigma^2 \bar{w} - \alpha X_t \right) (a - b X_t) - \frac{\sigma^2}{2} (a - b X_t)^2 \right] dt + \sigma \int_0^T (a - b X_t) dz \]

\[ = rT + \sigma^2 \int_0^T \left[ \left( \frac{\alpha}{\sigma^2} - \frac{1}{2} \bar{b} X_t^2 \right) + (ab - \frac{\alpha}{\sigma^2}a - c^2 \bar{w}b) X_t + \left( c^2 \bar{w} - \frac{1}{2}a \right) \right] dt + \sigma \int_0^T (a - b X_t) dz \]

In this case, we have

**Proposition 11.** The conformist expected return is

\[ E[R_T] = F(\alpha) X_0^2 + G(\alpha) X_0 + I(\alpha) \]

\[ F(\alpha) > 0 \]

Proof: see the Appendix\textsuperscript{10}.

First, note that when the world is predictable, the conformist’s expected return \( E[R_T] \) is a quadratic function of the initial mis-pricing \( X_0 \). Since the sign of the quadratic coefficient \( F(\alpha) \) is positive, larger subjective mis-pricing implies higher expected returns and vice versa. Note that the positive sign of the coefficient \( J(\alpha) \) does not depend on the agent’s beliefs through the speed \( \alpha \) and it does not depend on the risk aversion parameter \( c \).

\textsuperscript{10} The expected return cannot be obtained in closed form for an arbitrary values of \( c \). However, the expectation can be obtained for specific parameter values. For example, in a "typical case" of risk aversion parameter \( c = 2 \), we have:

\[ F(\alpha) = \frac{\alpha}{4 \sigma^2} \left[ \frac{4}{9} e^{-\alpha T} \left( \alpha T + \ln \left( \frac{3 + e^{-\alpha T}}{4} \right) \right) - 2 e^{-\alpha T} \left( \frac{1}{3} + e^{-\alpha T} - \frac{1}{4} e^{-2 \alpha T} + \frac{3}{4} \right) \right] \]

\[ G(\alpha) = 4 \bar{w} \left[ \frac{3 e^{-\frac{2 \alpha T}{3}}}{3 + e^{-\alpha T}} + \frac{1}{2} e^{-\alpha T} + \frac{1}{\sqrt{3}} e^{-\alpha T} \left[ \tan^{-1} \left( \frac{1}{\sqrt{3}} \right) - \tan^{-1} \left( \frac{1}{\sqrt{3}} e^{-\frac{T}{3}} \right) \right] - 1 \right] \]

\[ I(\alpha) = \left( r + 8 \sigma^2 \bar{w}^2 \right) T + \frac{1}{2} \left( \frac{3}{8} - \frac{1}{9} e^{-2 \alpha T} \right) \alpha T - \left( \frac{6 \sigma^2 \bar{w}^2}{\alpha} - \frac{1}{16} \right) \left[ 1 - e^{-\alpha T} \right] + \left[ 1 - \frac{1}{9} e^{-2 \alpha T} \right] \ln \left( \frac{3 + e^{-\alpha T}}{4} \right) + \frac{1}{32} e^{-2 \alpha T} + \frac{1}{4} e^{-3 \alpha T} - \frac{3}{32} \]


B.2. The Rebel.

In the predictable world, the rebels by definition ignore the predictability when they optimize their portfolios. We substitute the optimal $w$ from Proposition 2, so that

$$w = \bar{w}$$

However, the rebel’s terminal wealth is also path-dependent due to the trend-reversion of the true log-price process. The return is

$$R_T = rT + \int_0^T [(c^2\sigma^2\bar{w} - \alpha X_t)\bar{w} - \frac{\sigma^2}{2}\bar{w}^2] dt + \bar{w}\sigma \int_0^T dz$$

We have

Proposition 12. The rebel’s return is

$$R_T = [r + (c^2 - \frac{1}{2})\sigma^2\bar{w}^2]T + \bar{w}[X_T - X_0]$$

Proof: in the Appendix.

Observe that the realized return is higher when the positive over-pricing increases for the realized path. Intuitively, the likely paths have negative drift with positive over-pricing, and increased overpricing corresponds to a positive drift, so that the rebel who optimizes with subjective positive expected return gets a higher realized return when the realized return is more positive.

Since $X_T$ is normally distributed (see above), we have

Proposition 13. The rebel’s return is normally distributed with a mean of:

$$E[R_T] = [r + (c^2 - \frac{1}{2})\sigma^2\bar{w}^2]T - \bar{w}(1 - e^{-\alpha T})X_0$$
The variance $Var[R_T]$ is

$$Var[R_T] = \bar{w}^2 \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha T})$$

so that finally, the risk-reward trade-off measured by the Sharpe ratio is

$$\frac{E[R_T] - rT}{\sigma[R_T]} = \frac{1}{\sqrt{\frac{1 - e^{-2\alpha T}}{2\alpha T}}} \frac{(1 - \frac{1}{2\alpha^2})(\gamma + \frac{\alpha^2}{2} - r)T - (1 - e^{-\alpha T})X_0}{\sigma T}$$

Note that the Sharpe ratio is lower with higher positive over-pricing because the rebel is more out of line, with positive exposure when he would be shorting more if he had the correct specification. However, the Sharpe ratio increases as the under-pricing becomes more negative.

As an example that features “reasonable” parameter values, consider $\alpha = 1$, along with the set of other parameter fused in the previous examples, (i.e. $T = 1, c = 2, r = 0.05, \sigma = 0.25, \gamma = 0.15$). If the subjective price is in line with the trend, the Sharpe ratio is $SR(X = 0) = 0.6986$. Increasing the subjective under-pricing to $X = -0.1$, the Sharpe ratio increases to $SR(X = -0.1) = 1.0832$.

**B.3. Performance Comparison.**

The rebel’s expected return is linear in the current mis-pricing $X_t$. This leads to asymmetric dependence of the rebel’s expected return on the subjective mis-pricing. The linear coefficient is negative, hence the expected return decreases with increasing over-pricing and it decreases with increasing under-pricing of the market. We have proved

$Proposition 14$. The expected return of the rebel increases with increasing true under-pricing and decreases with increasing true over-pricing.
Intuitively, increasing the over-pricing implies higher negative expected returns, but the rebels ignores the over-pricing and does not decrease the holding of the risky asset. As a result, as the over-pricing increases, the fixed constant fraction of the rebel’s portfolio earns a more negative expected return. Conversely, higher market under-pricing implies that the constant fraction of the rebel’s wealth invested in the market is earning a higher positive expected return, so the rebel’s expected return increases with increasing under-pricing.

In contrast, the conformist expected return increases with both increased under-pricing and increased over-pricing. This is due to the fact that the conformist expected return is a quadratic function of the current mis-pricing. Qualitatively, the conformist’s expected return exhibits symmetric dependence on the mis-pricing, because the conformist correctly exploits any mis-pricing. We can summarize our findings in Proposition 15. The expected return of the conformist increases with increasing mis-pricing, while the rebel’s expected return increases only with increasing under-pricing.

Intuitively, the conformist is correctly timing the market both when the price is too low and when it is too high, while the rebel believes that the market is unpredictable and is not timing it at all.

IV. PREDICTABLE WORLD, REBELS’ WORLD

Which world is more forgiving of the rebel? Is it more dangerous to be a sophisticated rebel, who believes in predictability, in a simple log-normal world, or to be a simplistic rebel in a predictable world? The answers have important implications about how model risk and mis-specified beliefs affect performance evaluation. We compare the worlds in terms of the rebel’s reward, risk and reward-risk trade-off. Average historic return is often used as an investment performance indicator. Thus, higher expected returns may help a rebel whose survival hinges on such performance
measures. e.g. a fund manager. Next, comparing the risk exposures of the two worlds’ rebels reveals which world offers a better chance to survive. Finally, the risk-reward ratio is often viewed as the most important measure of investment performance. We examine the properties of a static risk-reward measure like the Sharpe ratio to gauge the attractiveness of rebellion in the two different worlds.

A. Expected Returns

The over-specified rebel who believes in predictability when the world is log-normal has decreasing expected returns when either the over-pricing or the under-pricing is increasing. Intuitively, the rebel is wrongly timing the market when both the price appears to be too low and when it appears too high.

In contrast, the simplistic rebel, who believes in a constant mean market return in a predictable world with trend-reverting log-prices is not punished as severely, because his expected return decline is not symmetrical in the mis-pricing. The rebel’s expected return decreases when the market is over-priced, but it is rising when the market is under-priced. We conclude that in terms of reward, as measured by the expected return, it is more rewarding to be a simplistic rebel in a predictable world, than to believe in predictability in an unpredictable world.

B. Total Risk

Wrong beliefs cause a sub-optimal risk exposure of the rebel’s optimal portfolio in any world. Consider how the sub-optimal risk exposure is affected by the wrong beliefs in the different worlds. This comparison reveals which world offers the rebel a better chance to survive by avoiding ruin.

The simple log-normal world punishes the rebel who believes in predictability with higher total risk, regardless of the direction of the subjective mis-pricing. On the other
hand, the variance of the simplistic rebel in the predictable world is not affected by the extent, or sign of the true mis-pricing. Hence, the total risk of the simplistic rebel is limited. We conclude that in terms of total risk, as measured by the variance of return, it is safer to be a simplistic rebel in a predictable world, than to believe in predictability in an unpredictable world.

C. Reward-Risk Trade-off

From an investment performance measurement perspective, it is most important to compare the perils of rebellion in terms of the reward-risk tradeoffs.

When the market truly is predictable, the simplistic rebel believes that the market is not predictable and ignores the trend-reversion when optimizing their portfolio. The expected return suffers incrementally only when the market is over-priced, while the risk remains bounded. As a result, the reward-to-risk trade-off of the simplistic rebel inherits the features of the rebel’s expected return. In particular, the reward-to-risk trade-off actually improves with more true under-pricing.

In contrast, when the market truly is unpredictable, a rebel who believes in predictability suffers both from lower expected returns and higher risk. Moreover, as the rebel’s beliefs become more extreme in terms of subjective mispricing, the rebel’s expected return decreases and the variance of returns increases. In terms of reward-risk trade-off, the unpredictable world is not a good place for rebels.

We showed that in terms of reward-to-risk trade-off, as measured by the ratio of expected return to variance of return, it is much better to be a simplistic rebel in a predictable world, than to believe in predictability in an unpredictable world. Sophisticated rebels who depend on performance evaluation to stay in business are unlikely to survive very long.
V. CONCLUSION

We consider investing based on subjective and possibly incorrect beliefs about market predictability. The market is noisy and two types of agents use different model specifications to optimize their portfolios. There are conformists, who happen to believe in a self-fulfilling market consensus and “rebels” who have wrong beliefs about market predictability. We take an agnostic approach about the true market process and consider both a scenario when the world is truly predictable and the alternative when it is unpredictable.

First, we compare the optimal portfolio strategies of two types of agents. We show that an agent who believes in log-normality is always a locally contrarian trader, who buys more shares after the price goes down, and sells shares when the price goes up. In contrast, an agent who believes in price predictability acts as a momentum trader for a range of prices that correspond to a range of perceived market mis-pricing. Intuitively, the agent’s perceived opportunity set is changing with the price level, so the agent is ”timing” the market and buys more shares after the price goes up in order to reach the optimal allocation for the new opportunity set.

We characterize the empirically observable investment performance for the agents in terms of the first and second moments of the terminal wealth distribution. In particular, we also consider the Sharpe ratio, which is a popular measure of risk-reward tradeoff. We find that a rebel expected return can get a higher with more incorrect beliefs about predictability. We show that in terms of expected terminal wealth, it is more dangerous to be a sophisticated rebel in a simple world, than to be a simplistic rebel in a predictable world. The over-complicated rebel who believes in predictability when the world is log-normal has decreasing expected returns both when the over-pricing is increasing and when the under-pricing is increasing. Intuitively, the rebel is wrongly timing the market both when the price appears to
be too low and when it appears too high. Moreover, the total risk of the rebel is increasing when the subjective mis-pricing is increasing. In contrast, in a predictable world with trend-reverting market, the simplistic rebel who believes in a constant mean of the market return is not punished as severely, because the expected return declines only for over-priced markets and the total risk is bounded. As a result, the Sharpe ratio of the naive rebel is increasing for more extreme subjective beliefs in more severe market underpricing.
REFERENCES


APPENDIX

A. The Optimal Portfolio

The Bellman equation for the derived utility of wealth $J$ is simplified to the following equation for $Q$

$$0 = Q_t + (1-c^2)\left[r + \frac{1}{2c^2} A_1\right] Q + \left[\frac{1-c^2}{c^2} A_2 - \alpha(\mu - \gamma)\right] Q^{\mu} + \frac{1-c^2}{2c^2} A_3 \left(\frac{Q^{\mu}}{Q}\right)^2 + \frac{1}{2} \alpha^2 \sigma^2 Q^{\mu \mu}$$

subject to the boundary condition $Q(\mu, T) = 1$, where the constants are from the optimal portfolio weights implied by the Bellman optimization

$$A_1 = \frac{\mu^2 - 2 (r - \frac{\sigma^2}{2}) \mu + (r - \frac{\sigma^2}{2})^2}{\sigma^2}, A_2 = -\alpha(\mu + \frac{\sigma^2}{2} - r), A_3 = \alpha^2 \sigma^2$$

Setting $\tau \equiv T - t$ we look for a solution of the form

$$Q(\mu, \tau) = \exp\left[E(\tau) \mu^2 + D(\tau) \mu + C(\tau)\right]$$

and setting

$$\bar{w} = \gamma + \frac{\sigma^2}{2} - r$$

we have

$$w = \bar{w} - H$$

$$H \equiv \frac{\alpha Q^{\mu}}{c^2 Q} = \frac{\alpha}{c^2} \frac{(2E\mu + D)}{a} \rightarrow 0$$

We substitute $Q$ above to obtain a system of ODE for the coefficients $D, E$. We have

$$E' = \frac{1}{c^2} \left[2\alpha^2 \sigma^2 E^2 - 2\alpha E + \frac{1-c^2}{2\sigma^2}\right]$$

$$D' = \frac{1}{c^2} \left[2\alpha \sigma^2 E - 1\right] \left[\alpha D + \frac{(r - \frac{\sigma^2}{2}) + \frac{c^2 \left(\gamma + \frac{\sigma^2}{2} - r\right)}{\sigma^2}}{\sigma^2}\right] + \frac{1}{\sigma^2} \gamma$$

and solving this system of differential equations we get the expressions in the text.
B. The Sophisticated Rebel in a Simple World

We have, e.g. if we set $\alpha = 0$ in the mean-reverting specification, $dX = -\alpha X ds + \sigma dz$ for $0 \leq s \leq T$

$$dX = \sigma dz$$

$$X(s) \equiv p(s) - \gamma s$$

and using the constant $\bar{w} \equiv (\gamma + \frac{\sigma^2}{2} - r)/c^2\sigma^2$, the return is

$$R_T = rT + \sigma^2 \int_0^T \left[ c^2 \bar{w}w - \frac{1}{2}w^2 \right] ds + \sigma \int_0^T w dz$$

The rebel believes in predictability. We substitute the path-dependent optimal portfolio $w(s), 0 \leq s \leq T$, as in

$$w(s) = a(s) - b(s)X_s$$

$$a(s) = \bar{w} + \bar{w}(c^2 - 1)(1 - e^{-\frac{\gamma}{c}(T-s)}) \frac{1 - \frac{c-1}{c+1}e^{-\frac{2\gamma}{c}(T-s)}}{1 + \frac{c-1}{c+1}e^{-2\gamma c(T-s)}}$$

$$b(s) = \frac{\alpha}{c\sigma^2} \frac{1 - \frac{c-1}{c+1}e^{-\frac{2\gamma}{c}(T-s)}}{1 + \frac{c-1}{c+1}e^{-\frac{2\gamma}{c}(T-s)}}$$

and the return is

$$R_T = rT + \sigma^2 \int_0^T [c^2 \bar{w}(a - bX_s) - \frac{1}{2}(a - bX_s)^2] ds + \sigma \int_0^T (a - bX_s) dz$$

$$= rT + \sigma^2 \int_0^T \left[ -\frac{1}{2}b^2 X_s^2 + (ab - c^2 \bar{w}b)X_s + (c^2 \bar{w}a - \frac{1}{2}a^2) \right] ds + \sigma \int_0^T (a - bX_s) dz$$

B. 1. The Expected Return

We have

$$E[R_T] = rT + \sigma^2 \int_0^T \left[ -\frac{1}{2}b^2 E[X_s^2] + (ab - c^2 \bar{w}b)E[X_s] + (c^2 \bar{w}a - \frac{1}{2}a^2) \right] ds =$$

$$= -\frac{\sigma^2}{2} X_0^2 I_{b^2} + \sigma^2 X_0 I_{abb} + rT + \sigma^2 I_{aa^2} - \frac{\sigma^4}{2} I_{sb^2}$$

$$= J(\alpha) X_0^2 + K(\alpha) X_0 + L(\alpha)$$

$$J(\alpha) \equiv -\frac{\sigma^2}{2} I_{b^2}, K(\alpha) \equiv \sigma^2 I_{abb}, L(\alpha) \equiv rT + \sigma^2 I_{aa^2} - \frac{\sigma^4}{2} I_{sb^2}$$
where we have set
\[
I_{b2} \equiv \int_0^T b^2 ds, I_{ab} \equiv \int_0^T (ab - c^2\bar{w}b)ds, I_{a^2a} \equiv \int_0^T (c^2\bar{w}a - \frac{1}{2}a^2)ds, I_{sb2} \equiv \int_0^T sb^2 ds
\]
and solving these integrals produces the expressions in the text. Moreover
\[
J(\alpha) \equiv -\frac{\sigma^2}{2} \int_0^T b^2 ds < 0
\]
since the integrand is non-negative.

B. 2. Proof of Proposition 8

We compare the rebel’s maximum expected return, \((-K^2/4J) + L\) to the conformist expected return \(E[R_{T}^{Conf}] \equiv [r + (c^2 - \frac{1}{2})\sigma^2\bar{w}^2]T\) and we have
\[
-\frac{K^2}{4J} = \frac{1}{2}(c^2 - 1)^2\sigma^2\bar{w}^2T \left(\frac{1-e^{-\frac{c}{c+1}\alpha T}}{1+e^{-\frac{c}{c+1}\alpha T}}\right)^2 \cdot \frac{1-e^{-2\alpha T}}{1+e^{-2\alpha T}} \cdot \frac{2e^{-\alpha T}}{1+e^{-2\alpha T}}
\]
and
\[
L(T) - E[R_{T}^{Conf}] = -\frac{T^2}{4c^2}2\alpha^2 + \frac{c - 1}{2c^2}T\alpha + \frac{1}{2}(c^2 - 1)^2\sigma^2\bar{w}^2T - \frac{1}{2}\ln \left(\frac{1 + \frac{c-1}{c+1}e^{-\frac{c}{c+1}\alpha T}}{1 + \frac{c-1}{c+1}e^{-2\alpha T}}\right)
\]
Consider the three terms as \(\alpha \to 0\). First, note that \(\frac{1-e^{-\alpha T}}{1+e^{-\alpha T}} \to 1\) so that
\[
-\frac{K^2}{4J} \to \frac{1}{2}(c^2 - 1)^2\sigma^2\bar{w}^2T > 0
\]
Next,
\[
L(T) - E[R_{T}^{Conf}] \to 0
\]
Hence as \(\alpha \to 0\), the difference between the maximum rebel expected return and the conformist expected return
\[
\max E[R_{T}^{rebel}] - E[R_{T}^{Conf}] = \left(-\frac{K^2}{4J} + L\right) - E[R_{T}^{Conf}] \to \frac{1}{2}(c^2 - 1)^2\sigma^2\bar{w}^2T > 0
\]
B. 3. The Variance

We have
\[
\text{Var}[R_T] = \sigma^2 \text{Var} \left[ \int_0^T (c^2 \bar{w}w - \frac{1}{2}w^2) ds + \int_0^T zdz \right]
\]
\[
= \sigma^4 \text{Var} \left[ \int_0^T (c^2 \bar{w}w - \frac{1}{2}w^2) ds \right] + \sigma^2 \int_0^T E[w^2] ds
\]

We use a subscript for time, as in \( b_s^2 \equiv b^2(s) \) so that the optimal portfolio \( w_s \) is
\[
w_s = a_s - b_sX_s
\]
and we have
\[
w_s \sim N \left( \theta_s, \omega_s^2 \right)
\]
\[
\theta_s \equiv a_s - X_0b_s
\]
\[
\omega_s^2 \equiv \sigma_s^2 b_s^2
\]
and we define the zero-mean variable
\[
w_s^* \equiv w_s - \theta_s \sim N \left( 0, \omega_s^2 \right)
\]
and note that \( w_s^* \) does NOT depend on \( X_0 \) Now, using
\[
w_s = w_s^* + \theta_s
\]
we have
\[
R_T = rT + \sigma^2 \int_0^T \left[ c^2 \bar{w}(w_s^* + \theta_s) - \frac{1}{2}(w_s^* + \theta_s)^2 \right] ds + \sigma \int_0^T (w_s^* + \theta_s) dz
\]
\[
= rT + \sigma^2 \int_0^T (c^2 \bar{w}\theta_s - \frac{1}{2}\theta_s^2) ds + \sigma^2 \int_0^T \left[ (c^2 \bar{w} - \theta_s)w_s^* - \frac{1}{2}w_s^*ds \right] + \sigma \int_0^T (w_s^* + \theta_s) dz
\]
The variance
\[
\text{Var}[R_T] = \sigma^2 \text{Var} \left[ \int_0^T (w_s^* + \theta_s) ds + \sigma \int_0^T [(c^2 \bar{w} - \theta_s)w_s^* - \frac{1}{2}w_s^*ds] \right]
\]
\[
= \sigma^2 I_1 + \sigma^4 I_2
\]
where

$$I_1 \equiv \int_0^T E[(w^*_s + \theta_s)^2] ds = \int_0^T (\theta^2_s + 2\theta_s E[w^*_s] + E[w^*_s^2]) ds = \int_0^T (\theta^2_s + \omega^2_s) ds$$

$$f(\alpha) \equiv \int_0^T a^2_s ds + \sigma^2 \int_0^T sb^2_s ds$$

and

$$I_2 \equiv Var \left[ \int_0^T [(c^2 w - \theta_s)w^*_s - \frac{1}{2} w^*_s^2] ds \right] =$$

$$= 2\sigma^2 \int_0^T \left[ \int_0^u s[k_s b_s k_u b_u + \frac{\alpha^2}{4}(2s + u)b^2_s b^2_u] ds \right] du - \frac{\sigma^4}{4} \left( \int_0^T sb^2_s ds \right)^2$$

$$k_s \equiv c^2 w - \theta_s = c^2 w - a_s + X_0b_s$$

We are interested in the coefficients on $X_0$ so we group according to powers of $X_0$:

$$I_2 + \frac{\sigma^4}{4} \left( \int_0^T sb^2_s ds \right)^2 - \frac{\sigma^4}{2} \int_0^T b^2_u \left[ \int_0^u s(2s + u)b^2_s ds \right] du = 2\sigma^2 \int_0^T k_u b_u \left( \int_0^u s k_s b_s ds \right) du$$

$$= 2\sigma^2 c^2 \bar{w} \int_0^T b_u (c^2 \bar{w}_s - a_u) \left( \int_0^u sb_s ds \right) du +$$

$$+ 2\sigma^2 X_0 \int_0^T b_u \left[ \int_0^u s(c^2 \bar{w}_s - a_s)b_s ds \right] du + 2\sigma^2 X^2_0 \int_0^T b^2_u \left( \int_0^u sb^2_s ds \right) du$$

or

$$I_2 = \left( 2\sigma^2 \int_0^T b^2_u \left( \int_0^u sb^2_s ds \right) du \right) X^2_0 + \left( 2\sigma^2 \int_0^T b^2_u \left[ \int_0^u s(c^2 \bar{w}_s - a_s)b_s ds \right] du \right) X_0 + g(\alpha)$$

$$g(\alpha) \equiv 2\sigma^2 c^2 \bar{w} \int_0^T b_u (c^2 \bar{w}_s - a_u) \left( \int_0^u sb_s ds \right) du + \frac{\sigma^4}{2} \int_0^T b^2_u \left[ \int_0^u s(2s + u)b^2_s ds \right] du -$$

$$- \frac{\sigma^4}{4} \left( \int_0^T sb^2_s ds \right)^2$$

Hence, we get

$$Var[R_T] = \sigma^2 I_1 + \sigma^4 I_2 =$$

$$= \sigma^2 \left( \int_0^T b^2_s ds + 2\sigma^4 \int_0^T b^2_u \left[ \int_0^u sb^2_s ds \right] du \right) X^2_0 +$$

$$+ 2\sigma^2 \left( \int_0^T a_s b_s ds + \sigma^4 \int_0^T b^2_u \left[ \int_0^u s(c^2 \bar{w}_s - a_s)b_s ds \right] du \right) X_0 + Q(\alpha)$$
\[ Q(\alpha) \equiv \sigma^2 f(\alpha) + \sigma^4 g(\alpha) \]

\[
\text{Var}[R_T] = M(\alpha)X_0^2 + 2\sigma^2 N(\alpha)X_0 + Q(\alpha)
\]

\[
M(\alpha) \equiv \sigma^2 \left( \int_0^T b_s^2 ds + 2\sigma^4 \int_0^T \left[ \int_0^u s b_s^2 ds \right] du \right) > 0
\]

where the last inequality is true because all integrands are positive.

**C. The Conformist in a Predictable World**

In the predictable world, we have \( \alpha > 0 \) in the mean-reverting specification,

\[
dX = -\alpha X ds + \sigma dz
\]

\[
X(s) \equiv p(s) - \gamma s
\]

for \( 0 \leq s \leq T \) and using the constant \( \bar{w} \equiv (\gamma + \frac{\sigma^2}{2} - r)/c^2 \sigma^2 \), the return is

\[
R_T = rT + \int_0^T [(c^2 \sigma^2 \bar{w} - \alpha X_s) w - \frac{\sigma^2}{2} w^2] ds + \sigma \int_0^T w dz
\]

We consider the conformist’s terminal wealth for trending Ornstein-Uhlenbeck log-price. We substitute the path-dependent optimal portfolio

\[
w_s = a_s - b_s X_s
\]

The internal rate of return is

\[
R_T = rT + \int_0^T [(c^2 \sigma^2 \bar{w} - \alpha X_s) (a - b X_s) - \frac{\sigma^2}{2} (a - b X_s)^2] ds + \sigma \int_0^T (a - b X_s) dz
\]

\[
= rT + \sigma^2 \int_0^T [(\frac{\alpha}{\sigma^2}b - \frac{1}{2}b^2) X_s^2 + (ab - \frac{\alpha}{\sigma^2}a - c^2 \bar{w}b) X_s + (c^2 \bar{w}a - \frac{1}{2}a^2)] ds +
\]

\[+ \sigma \int_0^T (a - b X_s) dz \]
C. 1. The Expected Return

The expected return is

\[ E[R_T] = rT + \sigma^2 \int_0^T \left( \frac{\alpha}{\sigma^2} b - \frac{1}{2} b^2 \right) E[X_s^2] + (ab - \frac{\alpha}{\sigma^2} a - c^2 \bar{w} b) E[X_s] + (c^2 \bar{w} a - \frac{1}{2} a^2) \] ds

\[ = rT + \frac{\sigma^2}{2\alpha} I_{bb^2} + (X_0^2 - \frac{\sigma^2}{2\alpha}) I_{bb^2 e^{-2a}} + X_0 \sigma^2 I_{ab^e a} + \sigma^2 I_{a^2 a}

\]

\[ = F(\alpha)X_0^2 + G(\alpha)X_0 + I(\alpha)

\]

where we have set

\[ I_{bb^2} \equiv \int_0^T (ab - \frac{\sigma^2}{2} b^2) ds, I_{bb^2 e^{-2a}} \equiv \int_0^T (ab - \frac{\sigma^2}{2} b^2) e^{-2\alpha s} ds,

\]

\[ I_{ab^e a} \equiv \int_0^T (ab - \frac{\alpha}{\sigma^2} a - c^2 \bar{w} b) e^{-\alpha s} ds, I_{a^2 a} \equiv \int_0^T (c^2 \bar{w} a - \frac{1}{2} a^2) ds,

\]

and solving these integrals with \( c = 2 \) produces the expressions in the text.

C. 2. Proof of Proposition 11

Note that

\[ 0 < b(s) = \frac{\alpha}{c\sigma^2} \frac{1}{1 + \frac{c-1}{c+1} e^{-2\alpha (T-s)}} < \frac{\alpha}{c\sigma^2}

\]

so that

\[ \alpha b - \frac{\sigma^2}{2} b^2 = b \left( \alpha - \frac{\sigma^2}{2} b \right) > 0\]

and hence the integrand in \( F(\alpha) \equiv \int_0^T (ab - \frac{\sigma^2}{2} b^2) e^{-2\alpha s} ds \) is positive, which implies \( F(\alpha) > 0 \).

D. The Naive Rebel in a Predictable World

We consider the rebel’s terminal wealth for trending Ornstein-Uhlenbeck log-price.

The rebel’s terminal wealth is also path-dependent. We substitute \( w \) as in

\[ w = \bar{w} = \frac{\gamma + \frac{\sigma^2}{2} - r}{c^2 \sigma^2} \]
We use the representation of the Ornstein-Uhlenbeck integral as a difference between a Brownian Motion \( z \) with \( z(0) = 0 \) and an Ornstein-Uhlenbeck process. The return is

\[
R_T = rT + \int_0^T [(c^2 \sigma^2 \bar{w} - \alpha X_s) \bar{w} - \frac{\sigma^2}{2} \bar{w}^2] ds + \bar{w} \sigma \int_0^T dz
\]

\[= [r + (c^2 - \frac{1}{2}) \sigma^2 \bar{w}^2]T + \bar{w}[X_T - X_0]\]

Hence the return is normally distributed with a mean of

\[E[R_T] = [r + (c^2 - \frac{1}{2}) \sigma^2 \bar{w}^2]T - \bar{w}(1 - e^{-\alpha T})X_0\]

and the variance is

\[Var[R_T] = \bar{w}^2 \sigma^2 \alpha (1 - e^{-2\alpha T})\]

so that finally, the Sharpe Ratio is

\[
\frac{E[R_T] - rT}{\sigma[R_T]} = \sqrt{\frac{2\alpha}{1 - e^{-2\alpha T}}(1 - \frac{1}{2c^2}) \frac{\gamma + \frac{\sigma^2}{2} - r}{\sigma}T - \frac{\sqrt{2\alpha}}{\sigma} \sqrt{\frac{1 - e^{-\alpha T}}{1 + e^{-\alpha T}X_0}}}
\]
Path-dependent Portfolio as a Function of the Predictable Specification Parameters

Figure 1

Path-dependent weight $w = a - bX$

- **Under-Priced** $X = 10\%$
- **No Mis-Pricing** $X = 0$
- **Over-Priced** $X = 10\%$
Rebel Expected Return as a Function of Subjective Mis-pricing

Figure 2

- Faster Reversion ($\alpha = 1.5$)
- Slower Reversion ($\alpha = 0.5$)
- Base Case Reversion ($\alpha = 1.0$)

Higher Expected Return

More Under-pricing

$X_{max}$

Expected Return

Mis-Pricing $X$