DURATION, CONVEXITY AND HIGHER ORDER HEDGING (REVISITED)

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ABSTRACT

Here the concepts of Duration and Convexity are studied when the term structure at a single point in time generally cannot be summarized by a finite number of state variables. Hence it is unclear whether calculating Duration and Convexity from partial derivatives makes sense. In this paper definitions of Duration and Convexity are provided that circumvent this problem and consistency with traditional measures is shown. The information required to compute Duration as defined in this paper consists of the term structure and the volatility of zero-coupon bonds. Convexity additionally requires a model of how this volatility will change over time. Schemes for calculating Duration and Convexity in practice are provided.

I. INTRODUCTION

This paper reexamines the concepts of Duration and Convexity within modern day term structure theory. The framework adopted to characterize interest rate dynamics is that of Heath-Jarrow-Morton (1992) (hereafter HJM (1992)). The reason for this framework choice is two-fold. First, and of primary motivation, is the fact that the HJM framework generically results in a path-dependent term structure.[1] Hence there is no guarantee that at a fixed point in time the term structure is a function of a finite number of state variables observable at that time. This is required for existing definitions of Duration and Convexity. Second, as we will show in the paper, Duration is closely related to the forward rate volatility structure (the volatility of all forward rate of different maturity) and once specified the computation of Duration is almost immediate.

The original notion of Duration founded by Macaulay (1938) is that the Duration of a bond is the present value weighted average of the times to all cash flows generated by the bond over the remainder of its life.[2] Macaulay proceeds to compute present values by discounting using the yield-to-maturity of the bond which is consistent with assuming a flat yield curve. For ease of exposition throughout this paper we consider all interest rates quoted in the form of annualized continuously compounded rates. Consequently the resulting Macaulay Duration, denoted D_{Mac}, of a bond with N remaining cash flows can be expressed as $D_{Mac} = \sum_{i=1}^{N} \tau_i C_i e^{-r\tau_i} \Big/ \sum_{i=1}^{N} C_i e^{-r\tau_i}$ where all yields of different maturity equal r and τ_i is the time until cash flow C_i occurs. Observing that the price of this bond, denoted B, can be computed via $B = \sum_{i=1}^{N} C_i e^{-r\tau_i}$ it is immediately apparent that $-\frac{1}{B}\frac{\partial B}{\partial r} = D_{Mac}$ which gives Hick's (1939) interpretation that Duration is a measure of a bond's sensitivity to interest rate changes.[3] This derivative based expression for Duration is now frequently

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1 Note that all term structure models constructed by considering a set of state variables that follow a joint-Markov diffusion process and having the term structure at time t be a function of these state variables at time t are also captured by the HJM (1992) framework. These term structure models are referred to as path-independent.

2 This is the essence of what is being described in Macaulay (1938) pages 46 to 48.

3 Hicks (1939, page 186) used the term "average period" instead of Duration and derived his result by computing the elasticity of the price with respect to the discount factor $\frac{1}{(1+YTM)}$ where YTM is the yield to maturity of the bond.

called "modified Duration".[4]

Fisher-Weil (1971) considered the use of the Duration concept in a risk management (immunization) setting. His measure of Duration mathematically captures Macaulay's original description of Duration without the implicit assumption of a flat (constant) yield curve. Letting $d(\tau)$ represent the present value of one dollar to be received τ years in the future Fisher-Weil Duration, denoted D_{FW}, can be expressed as $D_{FW} = \sum_{i=1}^{N} \tau_i C_i d(\tau_i) \Big/ \sum_{i=1}^{N} C_i d(\tau_i)$ where there are no shape restrictions on $d(\tau)$. To perform the immunization strategy Fisher-Weil have to assume some dynamics of the term structure. In particular they allow for an arbitrary forward rate curve to depict the current term structure and future forward rate curves take the form of the current forward rate curve that have undergone a parallel shift up or down. Without loss of generality we can depict this behavior by expressing all forward rate curves over time in the form of $r + \delta(\tau)$ where r is the short rate prevailing at the time and $\delta(\tau)$ is a deterministic function measuring the difference between the current forward rate curve and the current short rate. Observing that the price of a bond can be computed via $B = \sum_{i=1}^{N} C_i d(\tau_i)$ where $d(\tau) = \exp\left\{-\int_0^{\tau} r + \delta(v)\, dv\right\}$, it is again apparent that $-\frac{1}{B}\frac{\partial B}{\partial r} = D_{FW}$. That is, the modified Duration representation also holds for Fisher-Weil Duration.

Cox-Ingersoll-Ross (1979) (hereafter CIR (1979)) is the first work that considers the concept of Duration within the context of modern day term structure models. The term structure framework adopted is one where the short rate follows a time-homogeneous univariate Markov diffusion in the equivalent risk-neutral economy.[5] Classic examples of such models are those of Vasicek (1977) and Cox-Ingersoll-Ross (1985). This structure is enough to guarantee that at time t the present value of one dollar to be received on date T in the future can be expressed as a function of the short rate r at time t and time to maturity τ only and hence denoted $d(r,\tau)$. The Cox-Ingersoll-Ross measure of

[4] In most current text books there is a difference between Macaulay Duration and Modified Duration. Fisher (1966) and Ingersoll-Skelton-Weil (1978), amongst others, have shown this difference disappears when expressing interest rates in the form of continuous compounding.

[5] The term "equivalent risk-neutral economy" refers to the description of all processes under the equivalent risk-neutral probability measure in a complete bond market.

Duration for a coupon bond is "the [time to] maturity of the discount bond with the same [basis] risk" (CIR (1979) p. 56) where the Basis Risk of a bond with price B refers to $\frac{\partial B}{\partial r}\Big/B$. That is CIR Duration is defined as that number D_{CIR} which solves $\frac{\partial d(r,D_{CIR})}{\partial r}\Big/d(r,D_{CIR}) = \frac{\partial B}{\partial r}\Big/B$. The motivation for this definition of Duration stems from the observation that Basis Risk emerges from Itô's lemma as "the relative change in the price of a bond attributable to an unexpected shift in the spot rate" (CIR (1979) p.54). Consequently a discount bond with time-to-maturity D_{CIR} has the same exposure to movements in the short rate as the coupon bond with price B.

All of the above concepts of Duration have been related to $\frac{\partial B}{\partial r}\Big/B$ which can be interpreted as a measure of first order sensitivity to interest rate changes. This has further motivated consideration of $\frac{\partial^2 B}{\partial r^2}\Big/B$ which is called "Convexity" and interpreted as a second order sensitivity measure. However what if bond prices at time t cannot be represented as a function of the short rate r at time t alone as is generally the case in the HJM (1992) framework?[6] What is meant by Duration and Convexity in this case if we typically think of them as being related to $\frac{\partial B}{\partial r}\Big/B$ and $\frac{\partial^2 B}{\partial r^2}\Big/B$? Should we restrict our attention to the case where bond prices at time t are a function of the short rate only?[7] If we do this then we are losing the richness of HJM framework and further we are in fact reverting back to the CIR (1979) framework so nothing new has been achieved. Should we extend the ideas of Duration and Convexity by considering the sensitivity of a bond price with respect to every single interest rate on the yield curve, that is $\frac{\partial B}{\partial y(t,T_1)}\Big/B$ and $\frac{\partial^2 B}{\partial y(t,T_1)\,\partial y(t,T_2)}\Big/B$ for all possible maturity dates T_1 and T_2? This is a distinct possibility and now the concepts of Duration and Convexity become a vector and a square matrix if we restrict attention to several choices of T_1 and T_2 only. This idea is in the spirit of the key-rate Duration concept proposed by Ho (1992)

6 We are referring to the result of the HJM framework that generically the term structure is path-dependent. Consequently even if there is only one Brownian motion introducing uncertainty into the bond market then the entire term structure at time t must be considered relevant information. Consequently the short rate alone at time t cannot summarize the information imbedded in the term structure at that time and even for a multi-factor CIR (1979) type framework where the term structure can be represented as a function of a finite number of state variables the issues that will be discussed above still apply.

7 This is essentially what was considered in Au-Thurston (1995).

and the research by Nawalkha (1995)[8]. The downside of this approach is that generally the Duration vector and the Convexity matrix are infinite dimensional and we only have ad hoc guidance regarding how to construct a reasonable finite approximation of them.

The above issues motivate the present paper where extensions of the Duration and Convexity concepts are provided that overcome the limitations of thinking about them as partial derivatives. These extensions are obtained by considering their use in an interest rate risk management setting where Duration matching of assets and liabilities is used to mitigate interest rate risk over a short time period and the addition of a Convexity match reduces the frequency of re-balancing in order to maintain the Duration based hedge. Like the typical measure of Duration, the measure provided in this paper is a single number. However surprisingly the resulting Convexity measure is two-dimensional. A reconciliation between these alternative measures of Duration and Convexity show that they indeed collapse to the typical measures under appropriate circumstances. In order to calculate Duration and Convexity as proposed in this paper requires the term structure, the volatility of the term structure, and a description of how this volatility changes over time (required only for Convexity). Very little structure has been imposed in this paper regarding what these three components should look like[9] and consequently can be left to the users discretion. However for practical purposes a simple scheme for computing Duration and Convexity is provided and a numerical example given.

The above discussion has revolved around Duration and Convexity when considering changes in a single interest rate *r*. Two natural extension are apparent. First, if we think of Duration and Convexity as a first and second order hedge respectively, what about higher order hedging? Second, what if we want to consider a multi-factor setting where there are two fundamental interest

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[8] The above comment does not do research of Ho (1992) and Nawalkha (1995) justice since their approachs do allow for multiple Brownian motions which has advantages. We will also consider the case of multiple Brownian motions in Section V of this paper.

[9] The main restriction is that the term structure evolves as in the HJM (1992) framework and the term structure's volatility evolution can be described by a stochastic differential equation with the same Brownian motions that introduce uncertainty into the bond market.

rates driving the evolution of the term structure? The final part of this paper extends on the work described above to accommodate both of these issues.

The paper proceeds as follows. In Section II we provide the term structure framework and describe what we mean by "Basis Risk". Section III develops the concept of Duration in the given term structure framework presuming one underlying Brownian motion introduces all uncertainty in the bond market. Consistency with prior Duration measures is shown and a simple scheme for computing Duration in practice is provided. Section IV is the analogue of Section III for the measure of Convexity. Extensions to higher order hedging and the consideration of multiple Brownian motions is presented in Section V. Section VI summarizes and concludes.

II. THE FRAMEWORK

In this paper we consider asset and liability portfolios comprising of non-random, default-free cash flows where uncertainty is introduced via the random evolution of interest rates. Consequently the evolution of such portfolios is related to the evolution of the term structure. The term structure framework adopted is that of HJM (1992) with an arbitrary, but finite, number of Brownian motions introducing uncertainty into the bond market. Even though the sections immediately to follow only consider the case of one Brownian motion for the intuitive development of Duration and Convexity, we consider the case of multiple Brownian motions here as the last section of this paper provides extensions.

Let $P(t,T)$ denote the price of a one dollar face value, default free, zero coupon bond at time t which will mature at time T. Such a bond is referred to as a pure discount bond. Characterizing the dynamics of the bond price process as a stochastic differential equation, the arbitrage-free evolution of all pure discount bonds can be written as[10]

[10] This representation can be obtained using the same arguments presented in Vasicek (1977) or HJM (1992).

$$\frac{dP(t,T)}{P(t,T)} = \left(r(t) + \sum_{k=1}^{K} \lambda_k(\omega_t,t)\, \Gamma_k(\omega_t,t,T) \right) dt + \sum_{k=1}^{K} \Gamma_k(\omega_t,t,T)\, dW_k(t) \tag{1}$$

where $r(t)$ is the instantaneous risk-free interest rate defined by the return on the instantaneously maturing bond at time t[11], $\lambda_k(\omega_t,t)$ is the market price per unit of risk associated with the uncertainty introduced by the *kth* Brownian motion $W_k(t)$, $\Gamma_k(\omega_t,t,T)$ is the volatility associated with the *kth* Brownian motion at time t for the pure discount bond with maturity date T, and ω_t represents the possibility of dependence on the realization of the term structure up to time t, that is $\omega_t \in \mathscr{F}_t$ where $\mathscr{F}_t$ represents any information available at time t generated by the random evolution of the term structure. Here we allow incremental Brownian motions $dW_i(t)$ and $dW_j(t)$ to be correlated over time with correlation structure $\rho(\omega_t,t)\,dt$.

To ensure that the above price process depicts the characteristics of bonds we have to ensure that the volatility of the instantaneously maturing bond is zero and that the bond price is one dollar at maturity. Heath-Jarrow-Morton (1992) achieve this by considering the evolution of forward rates instead of bond prices. The instantaneous forward rate at time t for date T, denoted $f(t,T)$, is defined through the relation $P(t,T) = \exp\left\{ -\int_t^T f(t,v)\,dv \right\}$. Characterizing the resulting uncertain evolution of the forward rate curve via a stochastic differential equation results in the following no-arbitrage evolution for forward rates:[12]

$$df(t,T) = \left(\sum_{k=1}^{K} \sum_{j=1}^{K} \rho_{k,j}(\omega_t,t)\, \gamma_k(\omega_t,t,T) \int_t^T \gamma_j(\omega_t,t,v)\,dv + \sum_{k=1}^{K} \lambda_k(\omega_t,t)\, \gamma_k(\omega_t,t,T) \right) dt + \sum_{k=1}^{K} \gamma_k(\omega_t,t,T)\, dW_k(t) \tag{2}$$

where the forward rate volatility structure for the *kth* Brownian motion, $\gamma_k(\omega_t,t,T)$, is related to bond

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11 The instantaneous risk-free rate can be defined by $r(t) = -\partial \ln\big(P(t,T)\big) / \partial T \,|_{T=t}$.

12 Here we allow for a correlation structure amongst the Brownian motions. HJM (1992) provide their representation for orthogonal Brownian motions which is equivalent to the above representation after rotation since we can always find an orthogonal set of Brownian motions that reconstructs a set of correlated Brownian motions. The only reason for allowing a correlation structure is that it may prove useful in an empirical implementation.

volatility via $\Gamma_k(\omega_t,t,T) = -\int_t^T \gamma_k(\omega_t,t,v)\,dv$. In this paper we will refer to both bond volatility and the forward rate volatility structure.

Now consider a portfolio at time t containing default-free cash flows C_i, for I = 1 to N, to be made at respective times T_i in the future. Denote the value of this portfolio at time t as $B(t)$ which can be computed using the term structure at time t via $B(t) = \sum_{i=1}^{N} C_i P(t,T_i)$. Given the arbitrage-free dynamics of the term structure in (1) the change in the value of this portfolio over the next instant in time is

$$\begin{aligned} dB(t) &= \sum_{i=1}^{N} C_i\, dP(t,T_i) \\ &= \sum_{i=1}^{N} C_i P(t,T_i) \left(r(t) + \sum_{k=1}^{K} \lambda_k(\omega_t,t)\, \Gamma_k(\omega_t,t,T_i) \right) dt + \sum_{i=1}^{N} C_i P(t,T_i) \left(\sum_{k=1}^{K} \Gamma_k(\omega_t,t,T_i)\, dW_k(t) \right). \end{aligned}$$

Using the fact that $B(t) = \sum_{i=1}^{N} C_i P(t,T_i)$ the above can be expressed as

$$\frac{dB(t)}{B(t)} = \left(r(t) + \sum_{k=1}^{K} \lambda_k(\omega_t,t)\, V_k(t) \right) dt + \sum_{k=1}^{K} V_k(t)\, dW_k(t) \tag{3}$$

where $$V_k(t) = \frac{\sum_{i=1}^{N} C_i P(t,T_i)\, \Gamma_k(\omega_t,t,T_i)}{\sum_{i=1}^{N} C_i P(t,T_i)}$$

and $V_k(t)$ is interpreted as the volatility of the portfolio associated with the *kth* Brownian motion.

In the existing Duration and Convexity literature the change in the value of a portfolio is considered relative to the change in a particular basis, typically some reference interest rate. To maintain consistency with this idea we will express changes in bond and portfolio prices relative to a set of "basis factors". Here we wish to be nondescript regarding which interest rates to use as the set of basis factors and in fact allow for the possibility of using quantities other than interest rates. This is achieved by considering a set of K basis factors, K being the number of Brownian motions introducing uncertainty in the bond market, where $x_i(t)$ is the value of the *ith* basis factor at time t and $x(t)$ denotes the vector of these factors. The only requirement is that the evolution of $x(t)$ can be characterized in the form

$$dx(t) = \Big(m(\omega_t,t) + \Psi(\omega_t,t)\ \lambda(\omega_t,t) \Big)\, dt \ + \ \Psi(\omega_t,t)\, dW(t) \tag{4}$$

where $W(t) = \big[W_1(t) \ \ W_2(t) \ \ \cdots \ \ W_K(t) \big]^{\top}$, $m(\omega_t,t)$ is the vector drift of $x(t)$ in the equivalent risk-neutral economy (that is, under the equivalent risk-neutral probability measure), $\lambda(\omega_t,t)$ is the vector of preference parameters with the *kth* element being $\lambda_k(\omega_t,t)$, and $\Psi(\omega_t,t)$ is an invertible matrix with the element in row *I* and column *k* denoted $\Psi_{i,k}(\omega_t,t)$ which is the volatility of $x_i(t)$ associated with the *kth* Brownian motion. Examples of basis factors that can be used include specific interest rates whose dynamics can be determined from the term structure's evolution in (1), the set of Brownian motions $W_k(t)$ for $k = 1, 2, \cdots, K$ (achieved by setting $\Psi(\omega_t,t) = \mathbf{I}$ and $m(\omega_t,t) = -\lambda(\omega_t,t)$), and in fact interest rates or the set of Brownian motions under any equivalent probability measure. To express the changes in value of a single bond, or a portfolio of riskless cash flows, observe that the transitions of the vector of Brownian motions $dW(t)$ can be expressed in terms of the transitions of the basis factors $dx(t)$ via

$$dW(t) = -\Big(\Psi(\omega_t,t)^{-1}\, m(\omega_t,t) + \lambda(\omega_t,t) \Big)\, dt \ + \ \Psi(\omega_t,t)^{-1}\, dx(t) .$$

Consequently the evolution of pure discount prices given in (1) can be expressed as

$$\frac{dP(t,T)}{P(t,T)} = \left(r(t) - \sum_{k=1}^{K} m_k(\omega_t,t)\, \xi_k(\omega_t,t,T) \right) dt \ + \ \sum_{k=1}^{K} \xi_k(\omega_t,t,T)\, dx_k(t) \tag{5}$$

where $\xi_k(\omega_t,t,T) = \sum_{j=1}^{K} \Psi_{j,k}(\omega_t,t)^{-1}\ \Gamma_j(\omega_t,t,T)$ with $\Psi_{i,k}(\omega_t,t)^{-1}$ denoting the element in row *I* and column *k* of $\Psi(\omega_t,t)^{-1}$. Here we can interpret $\xi_k(\omega_t,t,T)$ as the sensitivity of zero coupon bond price $P(t,T)$

to a change in basis factor $x_k(t)$ over one instant in time, that is the "basis risk associated with factor $x_k(t)$". Similarly the evolution for the price of a portfolio given in (3) can be expressed as

$$\frac{dB(t)}{B(t)} = \left(r(t) - \sum_{k=1}^{K} m_k(\omega_t,t)\, \Phi_k(t) \right) dt + \sum_{k=1}^{K} \Phi_k(t)\, dx_k(t) \tag{6}$$

where $$\Phi_k(t) = \frac{\sum_{i=1}^{N} C_i P(t,T_i)\, \xi_k(\omega_t,t,T_i)}{\sum_{i=1}^{N} C_i P(t,T_i)}$$

with $\phi_k(t)$ the Basis Risk of the portfolio associated with factor $x_k(t)$. Further the above shows that a portfolio's Basis Risk is equal to the present value weighted average of the Basis Risks of the individual cash flows in the portfolio. The term Basis Risk here may seem different from that introduced by CIR(1979) however they will be reconciled in the next section.

III. DURATION

In the above characterization of interest rate dynamics note that the term structure at time t depends on how it has evolved from a previous point in time. It is only under very special circumstances that the term structure at time t can be recovered by observing a small number of state variables at that time. For example, consider the case of just one Brownian motion. Jeffrey (1995) has shown that only for very restrictive functional forms of the volatility structure can we express prices of pure discount bonds at time t as a function of the short rate at time t, time t and maturity T only; that is, of the functional form $P(r(t),t,T)$. In this case the value of a bond portfolio at time t containing non-random default free cash flows will depend on time t and $r(t)$ only and hence has the functional form $B(r(t),t)$. Now the analysis of CIR (1979) applies and the Duration of this portfolio can be obtained by computing $\frac{\partial B(r(t),t)}{\partial r(t)} \,/\, B(r(t),t)$. However, in general the term structure depends on the path that interest rates have taken over time so the value of a portfolio is generically of the functional form $B(\omega_t,t)$ where the short rate at time t is but one element of that realization;

that is $r(t) \in \omega_t$.[13] So how should we compute Basis Risk? Is it $\frac{\partial B(\omega_t,t)}{\partial r(t)}$ in the sense that all other elements of ω_t are held fixed? Should we now consider computing $\frac{\partial B(\omega_t,t)}{\partial x}$ for each $x \in \omega_t$?[14,15] Or is there another way of thinking about Duration altogether?

To define a concept of Duration consider a common use of the Duration measure, namely immunization, originally introduced by Fisher-Weil (1971). The standard immunization strategy, which under ideal circumstances guarantees the ability of making all payments in your liability portfolio, is I) to purchase an asset portfolio with the same present value as the liability portfolio, ii) match the current "Durations" of the asset and liability portfolios, and iii) at the end of every period liquidate the asset portfolio and purchase a new asset portfolio that again has the same "Duration" as the liability portfolio (referred to as re-immunization). For now the term "Duration" is undefined. To retain intuition in this section we consider term structure dynamics with only one source of uncertainty, one Brownian motion. Later, in section V, we will examine the more general case of an arbitrary number of Brownian motions.

Consider a liability portfolio at time t containing positive default-free liability payments C_i^L to be made at respective times T_i^L in the future. Denote the value of this liability portfolio at time t as $L(t)$ and from (6) it's dynamics can be represented as

$$\frac{dL(t)}{L(t)} = \left(r(t) - m(\omega_t,t)\,\Phi_L(t)\right)dt + \Phi_L(t)\,dx(t) \tag{7}$$

[13] For example, it simply may not be possible to summarize the information in the term structure at time t with a finite number of state variables in which case the entire term structure or the realization of interest rates over time is considered relevant information and are elements of ω_t.

[14] This idea can make sense for the case where a representation for the term structure exists such that it is a function of a finite number, say M, of state variables at time t even though M may exceed the number of Brownian motions introducing uncertainty in the bond market. Here the additional state variables are acting as sufficient statistics for the path-dependent information that the term structure depends upon. A classic example of such a path-dependent model is that of Ritchken-Sankarasubramanian (1995). However in the general case it is not possible that a finite number of state variables at one point in time can summarize the term structure making the calculation of $\partial B(\omega_t,t)/\partial x$ for each $x \in \omega_t$ infeasible.

[15] It is also instructive to compare this idea to the market practice of computing key-rate Durations, that is the sensitivities of a bond portfolio's value to changes in several key interest rates. This is achieved by perturbing the yield curve in a region around each key interest rate while keeping all regions around other key rates fixed. If the number of key-rates is large then key-rate Duration calculations are in the spirit of computing $\partial B(\omega_t,t) / \partial x$ for a large number of $x \in \omega_t$.

where $\Phi_L(t)$ is the Basis Risk of the liability portfolio with respect to basis $x(t)$. Similarly the dynamics for the value of an asset portfolio containing positive default-free cash receipts C_i^A to be received at respective times T_i^A in the future can be represented via

$$\frac{dA(t)}{A(t)} = \left(r(t) - m(\omega_t,t)\,\Phi_A(t)\right) dt + \Phi_A(t)\, dx(t) \tag{8}$$

where $A(t)$ represents the value of the asset portfolio at time t and $\Phi_A(t)$ is the Basis Risk of the asset portfolio with respect to basis $x(t)$.

The way to achieve the objective of the immunization strategy is to create a self-financing strategy using the above asset portfolio so that the dynamics of the liability portfolio are mimicked. The first part of this strategy requires that the chosen asset portfolio must have the same present value as the liability portfolio, that is $A(t) = L(t)$ which will be referred to as the "present value matching condition". This can be expressed as

$$\sum_{i=1}^{N_A} C_i^A P(t,T_i^A) = \sum_{i=1}^{N_L} C_i^L P(t,T_i^L) \tag{9}$$

where N_A and N_L respectively represent the number of cash flows remaining (beyond time t) in the asset and liability portfolios. The second part of the strategy requires that the asset portfolio mimics the dynamics of the liability portfolio over the next instant in time, that is $dA(t) = dL(t)$. This requires the corresponding coefficients of dt and $dx(t)$ to match across equations (7) and (8) . This is achieved when $\Phi_A(t) = \Phi_L(t)$ which means that the only thing required for the replication strategy is to match the Basis Risk of the asset portfolio to the Basis Risk of the liability portfolio; referred to as the "Basis Risk matching condition". Using (6) this condition can be expressed as

$$\frac{\sum_{i=1}^{N_A} C_i^A\, P(t,T_i^A)\, \xi(\omega_t,t,T_i^A)}{\sum_{i=1}^{N_A} C_i^A\, P(t,T_i^A)} = \frac{\sum_{i=1}^{N_L} C_i^L\, P(t,T_i^L)\, \xi(\omega_t,t,T_i^L)}{\sum_{i=1}^{N_L} C_i^L\, P(t,T_i^L)} \tag{10}$$

noting that any choice of basis will do. The last part of the strategy, that is re-immunization, requires that at the end of each instantaneous change in time the asset portfolio is liquidated and the money is reinvested in a new asset portfolio that again satisfies the Basis Risk matching condition, that is $\Phi_A(t+dt) = \Phi_L(t+dt)$. The present value matching condition at this new time is guaranteed since $A(t) = L(t)$ and $dA(t) = dL(t)$ by construction at time t. Consequently the above dynamically managed asset portfolio is a self-financing strategy that replicates the path of the liability portfolio's value.

The three parts of the above described self-financing strategy correspond to the three parts of the standard immunization strategy described earlier.[16] Consequently the Basis Risk matching condition $\Phi_A(t) = \Phi_L(t)$ can be used to provide a definition for Duration. The intention of Duration, as proposed by Macaulay, is that it provides a measure of the "longness" of a sequence of cash flows where the Duration of a single cash flow is unambiguously equal to the time to when that cash flow will occur (see Macaulay (1938) p. 43-44)[17]. To obtain a definition for Duration observe that the Basis Risk matching condition can be satisfied by placing a single zero-coupon bond in the asset portfolio; denote maturity date of this bond as $\overline{T}$. Since the immunization strategy calls for Durations of the asset and liability portfolios to equal we consequently have a definition for the Duration of the liability portfolio, namely $(\overline{T}-t)$. The present value matching condition, also required by the immunization strategy, is trivially obtained by holding the appropriate number of these zero coupon bonds. The Duration of an asset portfolio can be defined similarly by considering the single liability that immunizes the asset portfolio. Therefore the proposed definition of Duration is as follows:

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[16] Comparing the standard immunization strategy to the above self-financing strategy we see that the first part of both are identical and the third part of the immunization strategy achieves the same objective as the third part of the self-financing strategy. Consequently the concept of Duration must be obtained from the second part of the self-financing strategy.

[17] Macaulay was interested in the "longness" of a coupon bearing bond however since the discussion of this section refers to asset and liability portfolios we paraphrase using the term "sequence of cash flows".

Definition (DURATION):

Take as given at time t the term structure $P(t,T)$, the forward rate volatility structure $\gamma(\omega_t,t,T)$, and the volatility $\psi(\omega_t,t)$ of the chosen basis factor $x(t)$. The Duration of a portfolio containing positive default-free cash flows C_i to occur at respective times T_i, where $T_i > t$, is $(\overline{T}-t)$ where $\overline{T}$ solves[18]

$$\int_t^{\overline{T}} \frac{\gamma(\omega_t,t,v)}{\psi(\omega_t,t)}\, dv \;=\; \frac{\sum_{i=1}^N C_i\, P(t,T_i) \int_t^{T_i} \frac{\gamma(\omega_t,t,v)}{\psi(\omega_t,t)}\, dv}{\sum_{i=1}^N C_i P(t,T_i)} \tag{11}$$

noting that the solution for $\overline{T}$ is the same irrespective of the choice of basis.

For example, consider the case when the forward volatility structure takes the form $\gamma(\omega_t,t,T) = \sigma r(t)^\beta\, e^{-\kappa(T-t)}$ for positive constants σ, β and κ. This volatility structure is considered in Ritchken-Sankarasubramanian (1995), a special case of Cheyette (1992), and when $\beta = 0$ it corresponds to the volatility structure implied by the Vasicek (1977) model. Since Duration is invariant under the choice of basis, choose for simplicity the Brownian motion as the basis in which case $\psi(\omega_t,t) = 1$. Inserting the volatility structure into the above definition for Duration and solving for $(\overline{T}-t)$ implies the Duration measure

$$(\overline{T}-t) \;=\; \frac{-1}{\kappa} \ln\left(1 \;-\; \frac{\sum_{i=1}^N C_i\, P(t,T_i)\left(1 - e^{-\kappa(T_i-t)}\right)}{\sum_{i=1}^N C_i P(t,T_i)} \right).$$

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[18] To ensure that $\overline{T}$ exists it is enough to assume that all pure discount bond prices are strictly positive, that is $P(t,T) > 0$ for all $t \le T < \infty$, and the bond volatility structure $\Gamma(\omega_t,t,T)$ is continuous in the maturity dimension T for all $t \le T < \infty$. To see this let $w_i = C_i\, P(t,T_i) \Big/ \sum_{i=1}^N C_i\, P(t,T_i)$, observing $0 < w_i \le 1$ for all $i = 1, ..., N$ and $\sum_{i=1}^N w_i = 1$, so $\Gamma(\omega_t,t,\overline{T}) = \sum_{i=1}^N w_i \Gamma(\omega_t,t,T_i)$. Consequently $\Gamma_{\min} \le \Gamma(\omega_t,t,\overline{T}) \le \Gamma_{\max}$ where $\Gamma_{\min}$ and $\Gamma_{\max}$ are respectively the smallest and largest elements of $\left\{ \Gamma(\omega_t,t,T_i) \right\}_{i=1}^N$. Let $T_{\min}$ and $T_{\max}$ be the maturities associated with $\Gamma_{\min}$ and $\Gamma_{\max}$. Given the continuity of $\Gamma(\omega_t,t,T)$ in T implies $T_{\min} \le \overline{T} \le T_{\max}$ and the existence of $T_{\min}$ and $T_{\max}$ implies the existence of $\overline{T}$. If we further assume that the forward rate volatility structure $\gamma(\omega_t,t,T)$, where $\Gamma(\omega_t,t,T) = \int_t^T \gamma(\omega_t,t,v)\, dv$, is the same sign across maturity then $\Gamma(\omega_t,t,T)$ is also monotonic in T implying the uniqueness of $\overline{T}$.

An interesting observation is that the volatility of the short rate $\sigma r(t)^{\beta}$ does not enter the above Duration measure. The reason is immediately apparent from the definition of Duration where we could have set $\psi(\omega_t,t) = \sigma r(t)^{\beta}$ with the interpretation that the short rate $r(t)$ is the basis. An implication of this observation is to highlight the fact that the same Duration measure can result from a range of volatility structures, namely all those that are an arbitrary maturity independent scaling of each other.

From the above definition Duration can be interpreted as "the time-to-maturity of the zero coupon bond that has the same Basis Risk as the portfolio of cash flows". However it is important to realize that the matching of Basis Risks resulted from matching the transition in value of the zero coupon bond to that of the liability portfolio. Consequently it is perhaps more prudent to interpret the above definition of Duration as "the time-to-maturity of the zero coupon bond that *behaves like* the portfolio of cash flows". The term "behaves like" is meant in a local sense, that is the zero coupon bond and the portfolio have the same current value and will have identical realizations over the next instant in time.

A. A Scheme for Calculating Duration in Practice

From a practical point of view the above definition suggests an interesting scheme for determining the Duration of a portfolio at a particular point in time t. First observe that we have the freedom of arbitrarily choosing a basis which corresponds to specifying $m(\omega_t,t)$ and $\psi(\omega_t,t)$. If we set $m(\omega_t,t) = 0$ and $\psi(\omega_t,t) = 1$ [19] then this provides the convenient interpretation that the Basis Risk of a bond portfolio is the present value weighted average of the volatilities from each cash flow in the portfolio. Present value factors at time t can be obtained by estimating the yield curve at that time and the volatility at time t of a cash flow to occur at time T is equal to the bond volatility structure $\Gamma(\omega_t,t,T)$ which can be estimated from the time-series of historical term structure transitions. A reasonable

[19] This corresponds to choosing the Brownian motion that introduces uncertainty into the bond market in the equivalent risk-neutral economy as the basis. However as a practicality this is unimportant.

first-cut approximation to the bond volatility structure can be obtained by arguing that the volatility of a zero-coupon bond with time-to-maturity τ does not change significantly over a short period of time. This statement corresponds to modeling the bond volatility structure $\Gamma(\omega_t,t,T)$ purely as a function of time-to-maturity $\Gamma(\tau)$ and using recent historical data to estimate it. An estimate for $\Gamma(\tau)$ can be obtained by computing the sample standard deviation of log-bond price transitions for the zero-coupon bond with time-to-maturity τ[20]

$$\Gamma(\tau) = \sqrt{\frac{1}{\Delta t}\,\mathrm{VAR}\big[\Delta\ln\big(P(t,t+\tau)\big)\big]} \tag{12}$$

where $VAR[\,x\,]$ denotes the sample variance of x, Δt represents a small change in time such as one day, $P(t,t+\tau)$ represents the price at time t of a pure discount bond with time-to-maturity τ, and $\Delta\ln\big(P(t,t+\tau)\big) = \ln\big(P(t+\Delta t, t+\Delta t+\tau)\big) - \ln\big(P(t,t+\tau)\big)$ represents the change in the log of the price of a τ-year zero-coupon bond from one day to the next.

To demonstrate the calculation of Duration consider a 9 year, \$100 face value bond with a coupon rate of 5% per annum payable semi-annually on June 15, 1994. Further, for comparison, both the Macaulay and Fisher-Weil Duration measures are also computed. The first step toward computing Duration is to estimate the term structure on June 15, 1994. This can be achieved using

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[20] The evolution of bond prices driven by one Brownian motion can be obtain from (1) by setting K=1. Applying Itô's lemma to $\ln\big(P(t,T)\big)$ provides the evolution for the natural logarithm of bond prices which demonstrates that the volatility of $d\ln\big(P(t,T)\big)$ is the same as that of $\frac{dP(t,T)}{P(t,T)}$. Considering the evolution of $\ln\big(P(t,t+\tau)\big)$ with a fixed time-to-maturity τ instead of a fixed maturity date T we obtain

$$d\ln\big(P(t,t+\tau)\big) = \left(r(t) + \lambda(\omega_t,t)\,\Gamma(\omega_t,t,t+\tau) - \tfrac{1}{2}\Gamma(\omega_t,t,t+\tau)^2 + \frac{\partial\ln\big(P(t,t+\tau)\big)}{\partial\tau}\right)dt + \Gamma(\omega_t,t,t+\tau)\,dW(t)\ .$$

Now consider the case when $\Gamma(\omega_t,t,T)$ is of the form $\Gamma(T-t)$. Discretizing the above, squaring both sides and taking unconditional expectations of both sides provides

$$E\Big[\big(\Delta\ln\big(P(t,t+\tau)\big)\big)^2\Big] = \Gamma(\tau)^2\,\Delta t + o(\Delta t)$$

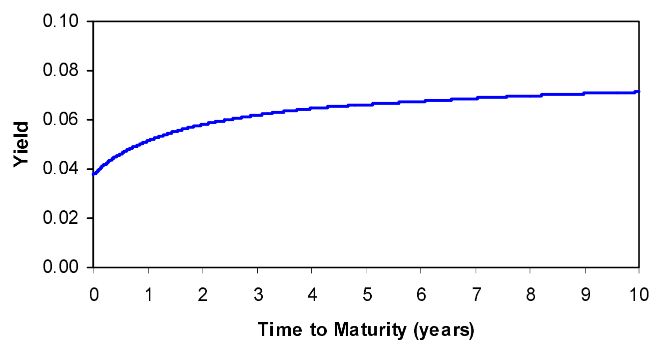
where Δt represents a fixed discrete time step and $\Delta\ln\big(P(t,t+\tau)\big) = \ln\big(P(t+\Delta t, t+\Delta t+\tau)\big) - \ln\big(P(t,t+\tau)\big)$. It is also true that

$$E\Big[\big(\Delta\ln\big(P(t,t+\tau)\big) - m(\tau)\,\Delta t\big)^2\Big] = \Gamma(\tau)^2\,\Delta t + o(\Delta t)$$

for any $m(\tau)$. Consequently letting $m(\tau)$ be the sample mean of observations $\Delta\ln\big(P(t,t+\tau)\big)$ and replacing the above expectation with the sample mean suggests that a first order approximation for $\Gamma(T-t)$ can be obtained from the sample variance of log-bond price transitions.

any of the existing flexible curve fitting procedures such as McCulloch (1971, 1975), Fama-Bliss (1987), and Linton-Mammen-Nielson-Tanggaard (2000). We choose the latter to estimate the yield curve $y(t,T)$ at time t and compute the discount function via the relationship $P(t,T) = e^{-y(t,T)\,\mathrm{x}\,(T-t)}$. The estimated yield curve on June 15, 1994 is given in Figure 1 below.

FIGURE 1
Yield Curve on June 15, 1994.



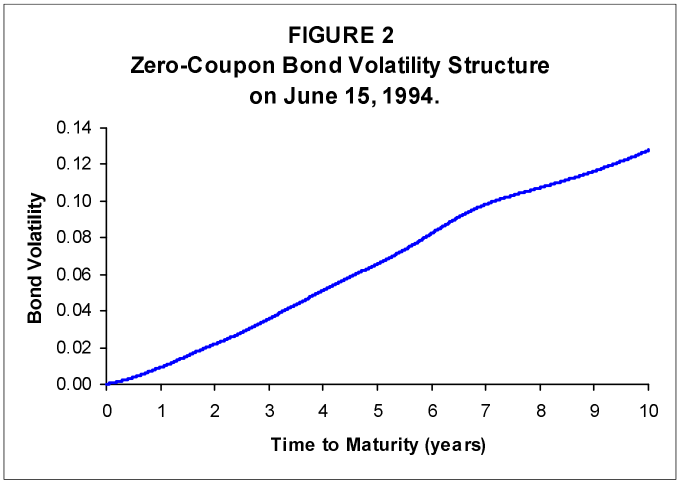
From this yield curve the present value of each cash flow generated by the bond is computed and summing these provides the bond's value of \$86.2937. At this point the above provides enough information to compute both the Fisher-Weil and Macaulay Duration measures:

$$\text{Fisher-Weil Duration} = \frac{\sum_{i=1}^{N} C_i\, P(t,T_i)\,(T_i - t)}{\sum_{i=1}^{N} C_i\, P(t,T_i)} = 7.17 \text{ years}$$

$$\text{Macaulay Duration} = \frac{\sum_{i=1}^{N} C_i\, e^{-\bar{y}\,\mathrm{x}\,(T_i - t)}\,(T_i - t)}{\sum_{i=1}^{N} C_i\, e^{-\bar{y}\,\mathrm{x}\,(T_i - t)}} = 7.20 \text{ years}$$

where $\bar{y}$ = 6.9631% p.a. continuously compounded. represents the yield-to-maturity of the coupon bond.

For the measure of Duration given in (11) the volatility structure of zero-coupon bonds is also required. Using six months of daily data just prior to June 15, 1994 volatility estimates for bonds with time-to-maturities from 0 to 10 years are calculated using (12) with $\Delta t = \frac{1}{250}$ representing one trading day. The resulting estimate for the bond volatility structure on June 15, 1994 is depicted in Figure 2 below.



From (3) the volatility of a portfolio of default-free cash flows is the present value weighted average of the volatilities from each cash flow in the portfolio and hence the coupon bond's volatility is

$$\begin{pmatrix} \text{coupon bond} \\ \text{volatility} \end{pmatrix} = \frac{\sum_{i=1}^{N} C_i \, P(t,T_i) \, \Gamma(\omega_t,t,T_i)}{\sum_{i=1}^{N} C_i \, P(t,T_i)} = 0.092777$$

remembering that here the coupon bond's volatility is a measure of the bond's Basis Risk. The Duration of this bond can now be obtained from Figure 2 by finding the time-to-maturity of the zero coupon bond with the same volatility, that is

$$\text{Duration} = 6.60 \ \text{years}.$$

For the example above the difference between the Macaulay and Fisher-Weil measures is small relative to the difference observed when using the Duration measure proposed in this paper. Whether this difference is economically or statistically significant remains an empirical question. However, before proceeding with such a study it should be noted that a caveat regarding the above is that the analysis assumes only one Brownian motion driving the dynamics of the entire term structure. If this is not the case, as suggested by the multi-factor term structure literature, Duration must be considered in a the multi-factor context presented in Section V.

B: Consistency With the Cox-Ingersoll-Ross (1979) Framework

The CIR framework is one where the short rate follows a univariate Markov diffusion[21] in the equivalent risk-neutral economy and the price of a pure discount bond with maturity date T is a function of the short rate $r(t)$ and time t only. That is, the dynamics of the short rate can be expressed as[22]

$$dr(t) \;=\; \Big(\theta(r(t),t) + \lambda(\omega_t,t)\,\sigma(r(t),t)\Big)\,dt \;+\; \sigma(r(t),t)\,dW(t)$$

where $\theta(r(t),t)$ is the drift of the short rate in the equivalent risk-neutral economy, $\sigma(r(t),t)$ is the volatility of the short rate, and the prices for all pure discount bonds takes the functional form $P(r(t),t,T)$. Applying Itô's lemma to $P(r(t),t,T)$ show that the bond volatility structure $\Gamma(\omega_t,t,T)$ takes the form $\frac{\partial P(r(t),t,T)}{\partial r(t)}\frac{\sigma(r(t),t)}{P(r(t),t,T)}$. Placing this form into the definition of Duration given in (11) and choosing the short rate as the basis, implying $\psi(\omega_t,t) = \sigma(r(t),t)$, states that Duration is $(\overline{T}-t)$ where $\overline{T}$ solves

$$\frac{\partial P(r(t),t,\overline{T})}{\partial r(t)} \Big/ P(r(t),t,\overline{T}) \;=\; \frac{\sum_{i=1}^{N} C_i\,P(r(t),t,T_i)\left(\frac{\partial P(r(t),t,T_i)}{\partial r(t)} \Big/ P(r(t),t,T_i)\right)}{\sum_{i=1}^{N} C_i P(r(t),t,T_i)} \tag{13}$$

where the left and right hand sides of the above are the Basis Risks, with respect to the short rate, of the desired zero coupon bond and the portfolio respectively. Observing that the right hand side of (13) is $\frac{\partial B(r(t),t)}{\partial r(t)} \big/ B(r(t),t)$, where $B(r(t),t) = \sum_{i=1}^{N} C_i P(r(t),t,T_i)$, we see that imposing the framework of CIR(1979) results in $\frac{\partial B(r(t),t)}{\partial r(t)} \big/ B(r(t),t)$ being a measure of Basis Risk for the portfolio.

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[21] Actually, only a time-homogeneous univariate Markov diffusion is considered however their arguments can clearly be extended to the non-homogeneous case discussed here.

[22] Note that it is only necessary to have the short rate process follow a Markov diffusion under the equivalent risk-neutral probability measure. This is the reason why the market price of risk can be of the form $\lambda(\omega_t,t)$.

This is identical to the measure of Basis Risk as defined in CIR(1979, page 54). Further CIR(1979, page 56) define the Duration of a portfolio to be the "[time to] maturity of a discount bond with the same [basis] risk". This statement is exactly the measure of Duration given in (13) above showing that the CIR (1979) concept of Duration is consistent with the definition of Duration given in this paper.

C. Modified Duration

The term Modified Duration is defined as the negative of the percentage change in the price B of a portfolio for a small change in a given interest rate r, that is $-\frac{\partial B}{\partial r}/B$. A common motivation for considering modified Duration stems from considering a portfolio's value at a fixed point in time as a function of an interest rate r and expressing the percentage change in the portfolio's value $\frac{\Delta B}{B}$ resulting from a discrete change in the level of the interest rate Δr via the first order Taylor expansion

$$\frac{\Delta B}{B} = \frac{1}{B}\left(\frac{\partial B}{\partial r}\Delta r + o\left((\Delta r)^2\right)\right).$$

The negative of the coefficient of the first order term Δr is Modified Duration which can be interpreted as a first order sensitivity measure to the change in interest rate levels.

The implicit assumption in the above is that the value of the portfolio at a fixed point in time is solely a function of r. In the context of Fisher-Weil the price at time t of a default-free portfolio of positive cash flows is computed via $B = \sum_{i=1}^{N} C_i \exp\left\{-\int_t^T r + \delta(v)\, dv\right\}$ where r is the short rate and $\delta(v)$ is a deterministic function that does not change through time. The Macaulay setting can be considered the special case of Fisher-Weil when $\delta(v) = 0$ for all v. In both of these cases the type of term structure transitions are limited, in particular it is assumed that the yield curve evolves through time by undergoing parallel shifts up or down. To consider a general term structure framework in which the value of a default-free portfolio at a fixed point in time is solely a function of r we must turn to the no-arbitrage framework considered in CIR(1979)[23]. Within this framework Basis Risk is identical to modified Duration. However if we wish to consider a term structure framework where the value of a default-free portfolio at a fixed point in time is also dependent on information other than the interest rate r[24] then the measure of Modified Duration needs to be extended. Observing that Basis Risk is identical to Modified Duration in the traditional settings suggests generalizing the term Modified Duration to be identical to Basis Risk as defined in the present paper, that is

$$\begin{pmatrix} \textit{Modified} \\ \textit{Duration} \end{pmatrix} = \frac{\sum_{i=1}^{N} C_i P(t,T_i) \frac{\Gamma(\omega_r,t,T_i)}{\psi(\omega_r,t)}}{\sum_{i=1}^{N} C_i P(t,T_i)}. \tag{14}$$

An alternative motivation for this extended definition of Modified Duration can be obtained by observing from (6) that a first order approximation for the percentage change in a portfolio's value as a result of a change in basis $x(t)$ can be written as[25]

$$\frac{dB(t)}{B(t)} = \Phi(t)\, dx(t)$$

where $\Phi(t)$ is computed via the modified Duration measure (14) above. Consequently this extension to the measure of Modified Duration is consistent with the original intent of Modified Duration, namely to provide a first order measure of a portfolio's sensitivity to a specified basis.

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23 It is interesting to note that parallel shifts in the yield curve cannot be sustained in an arbitrage-free setting; see Ingersoll-Skelton-Weil (1978). However the Fisher-Weil Duration measure can result from an arbitrage free term structure model that does not imply parallel term structure transitions. This is considered in CIR (1979) and further considered in the next sub-section.

24 As a reminder we are still only considering one underlying Brownian motion in the bond market and the statement regarding dependence on information other than r should be interpreted in the sense that r is not a sufficient statistic to summarize the entire term structure at a single point in time bringing us back to the original motivation of this paper.

25 This results by observing that $dx(t)$ is of order $\sqrt{dt}$ and hence the term $\left(r(t) - m(\omega_r,t)\,\Phi(t)\right) dt$ is of smaller order.

C: Macaulay and Fisher-Weil Duration Revisited

Cox-Ingersoll-Ross (1979) consider the issue of when the Fisher-Weil Duration measures can result within their framework. They show that the only term structure model generated by a time-homogeneous univariate Markov diffusion model for the short rate in the equivalent risk-neutral economy is when the drift and volatility coefficients of the short rate process are constants, that is $dr(t) = \mu\, dt + \sigma\, dW(t)$. Further the functional form for pure discount bonds is $P(r,(t),t,T)$ = $\exp\left\{-(T-t)\, r(t) - \frac{\mu}{2}(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3\right\}$.[26] For the case of Macaulay Duration where the forward rate curve is flat, the above term structure form shows that this can only occur when $\mu = 0$ and $\sigma = 0$, that is, when interest rates never change over time. These results suggest that the previous Duration measures are quite restrictive in the context of an immunization strategy. However are there other term structure models consistent with the Fisher-Weil and Macaulay Duration measures if one goes beyond the CIR framework?

To answer this question suppose the Duration of a portfolio of default-free cash flows is computed via the Fisher-Weil Duration measure

$$(\bar{T}-t) = \sum_{i=1}^{N} w_i \,\mathrm{x}\, (T_i - t) \quad \text{where} \quad w_i = \frac{C_i\, P(t,T_i)}{\sum_{i=1}^{N} C_i\, P(t,T_i)}$$

noting that no restrictions are place on the shape of the term structure at time t. Substituting the Fisher-Weil Duration calculation into the definition of Duration provided in this paper implies

$$\frac{\Gamma\left(\omega_t, t, t + \sum_{i=1}^{N} w_i \,\mathrm{x}\, (T_i - t)\right)}{\psi(\omega_t, t)} = \sum_{i=1}^{N} w_i \frac{\Gamma(\omega_t, t, T_i)}{\psi(\omega_t, t)}$$

where the chosen basis defines $\psi(\omega_t,t)$. At each fixed point in time t the bond volatility structure $\Gamma(\omega_t,t,T_i)$ possess the properties of a linear operator and consequently $\Gamma(\omega_t,t,T_i)$ must be linear in

[26] In the CIR framework, where only a time-homogeneous short rate process is considered, an arbitrary term structure shape is not consistent with the Fisher-Weil Duration measure. However if we allow the drift of the short rate to be a deterministic function of time then we can always calibrate the drift term to an arbitrary initial term structure and maintain Fisher-Weil Duration measure.

the third argument. Since $\Gamma(\omega_t,t,T_i) = -\int_t^T \gamma(\omega_t,t,v)\,dv$ the only possible forward rate volatility structures admitting the Fisher-Weil Duration measure are those independent of maturity, that is $\gamma(\omega_t,t,T) = \sigma(\omega_t,t)$.

Macaulay Duration can be considered as the special case of Fisher-Weil Duration when forward rate curves are always flat. Unfortunately for any volatility structure of the form $\gamma(\omega_t,t,T) = \sigma(\omega_t,t)$ the no-arbitrage dynamics of forward rates given by (2) implies that if a forward rate curve is initially flat it will not remain flat as time passes.[27] Consequently at best it can be said that Macaulay Duration is valid at time t if the forward rate curve happens to be flat at that time and if the forward rate volatility structure at time t is of the form $\gamma(\omega_t,t,T) = \sigma(\omega_t,t)$. However after time t the Macaulay Duration measure no longer applies.

IV. CONVEXITY

To perform the immunization strategy described in the previous section requires continual liquidation and reconstitution of the asset portfolio. In practice this is clearly not feasible and begs the question of how to reduce the number of times the asset portfolio has to be re-balanced. "Convexity matching" has been used in practice to handle this issue where the Convexity of a portfolio is a standardized measure of the second partial derivative of a portfolio's value B with respect to a particular interest rate r, in particular $\frac{\partial^2 B}{\partial r^2} \Big/ B$. Motivation for using Convexity for the purpose of reducing the frequency of re-balancing is obtained by extending the Taylor expansion motivation of Modified Duration. That is, consider a portfolio's value at a fixed point in time as a function of a particular interest rate r and expressing the change in the portfolio's value ΔB resulting from a discrete change in the level of the interest rate Δr via

$$\frac{\Delta B}{B} = \frac{1}{B}\left(\frac{\partial B}{\partial r}\Delta r + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}(\Delta r)^2 + o\big((\Delta r)^3\big)\right).$$

[27] This is consistent with the finding of Ingersoll-Skelton-Weil (1978) where they show that Macaulay Duration admits arbitrage.

This suggests the following strategy. First match the modified Duration of an asset portfolio to that of the liability portfolio. This provides a first order approximation for ensuring the values of the asset and liability portfolios respond similarly to a discrete change in the interest rate r. Second match the Convexities of the asset and liability portfolios. This should improve the similarity of value change behavior between the asset and liability portfolios when a change in the interest rate level occurs. The fact that asset and liability portfolios are now more closely aligned in terms of their behavior to interest rate changes suggests that less frequent re-immunization is required in practice.

The implicit assumptions with the above argument are I) the value of the portfolio at a fixed point in time is solely a function of r, and ii) interest rate changes are the dominant cause of portfolio value changes. The latter assumption is evident after realizing that interest rates only change with the passage of time but time itself also causes the portfolio to change value since the cash flows within the portfolio are closer to maturity. Both of these assumptions are violated in the context of the term structure framework presented in section II since a portfolio's value at time t is more than a function of a single interest rate and for time horizons greater than or equal to one instant time dt, time also significantly affects a portfolio's value.[28] So now we are left with the question "how can we think about Convexity if we want to relax these two assumptions"?

To provide a definition for Convexity consider the intention of its use within an immunization strategy, namely reducing the number of times re-immunization has to take place. In the continuous time framework adopted in this paper re-immunization has to be conducted at the end of every period where one period is an instant in time dt. What is proposed in this section is to halve the number of times that re-immunization has to take place and define Convexity based on what is required to achieve this goal. Of course theoretically this means re-immunization has to be conducted at time intervals of $2dt$ as opposed to dt so the frequency of re-immunization is still impractical. However the intention is that the inclusion of Convexity matching in an immunization strategy will

[28] It is also true that in the CIR (1979) framework the passage of time significantly affects a portfolio's value for horizons greater than or equal to an instant in time dt. We will consider the CIR (1979) setting in subsection B to follow.

carry over in practice by implying that less frequent re-immunization is needed over reasonably small but finite time intervals.

To again retain intuition this section considers term structure dynamics with only one source of uncertainty and the extension to an arbitrary number of Brownian motions is considered in section V. To motivate a definition for Convexity again consider the Duration immunization strategy presented in section III. To perform Duration immunization both the present value matching condition $A(t) = L(t)$ and the Basis Risk matching condition $\Phi_A(t) = \Phi_L(t)$ must be satisfied. Together these conditions imply $A(t+dt) = L(t+dt)$. Consequently to ensure the value of the asset portfolio is equal to that of the liability portfolio after two periods have transpired, that is $A(t+2dt) = L(t+2dt)$, we require the Basis Risk matching condition to be satisfied at time $t+dt$, that is $\Phi_A(t+dt) = \Phi_L(t+dt)$. To solve this problem the dynamics for the Basis Risk of zero coupon bonds $\xi(\omega_t,t,T)$ is required and using (6) the dynamics for any portfolio's Basis Risk can be computed. Therefore suppose the evolution of $\xi(\omega_t,t,T)$ can be characterized by the following stochastic differential equation

$$d\xi(\omega_t,t,T) = \Big(\alpha(\omega_t,t,T) + \lambda(\omega_t,t)\,\eta(\omega_t,t,T)\,\psi(\omega_t,t)\Big)\,dt + \eta(\omega_t,t,T)\,\psi(\omega_t,t)\,dW(t) \qquad (15)$$

where $\alpha(\omega_t,t,T)$ is the drift of Basis Risk in the equivalent risk-neutral economy, $\eta(\omega_t,t,T)\,\psi(\omega_t,t)$ is the volatility of Basis Risk, and $W(t)$ is the same Brownian motion that introduces uncertainty into the bond market. The reason for expressing the volatility of Basis Risk in the form $\eta(\omega_t,t,T)\,\psi(\omega_t,t)$ is so we can interpret $\eta(\omega_t,t,T)$ to be the Basis Risk of $\xi(\omega_t,t,T)$. This interpretation is obtained by expressing the above dynamics in the form

$$d\xi(\omega_t,t,T) = \Big(\alpha(\omega_t,t,T) - m(\omega_t,t)\,\eta(\omega_t,t,T)\Big)\,dt + \eta(\omega_t,t,T)\,dx(t) \qquad (16)$$

interpreting the coefficient of $dx(t)$ as a measure of $\xi(\omega_t,t,T)$'s sensitivity to $x(t)$. Since

$\xi(\omega_t,t,T) = \frac{1}{\psi(\omega_t,t)} \int_t^T \gamma(\omega_t,t,T)\,dv$ and $\psi(\omega_t,t)$ is determined from the specification of the basis, it is perhaps more natural to model the evolution of $\gamma(\omega_t,t,T)$ and then determine the implied dynamics of $\xi(\omega_t,t,T)$.[29]

Given that the Duration immunization conditions hold the requirement that $\Phi_A(t+dt) = \Phi_L(t+dt)$ is equivalent to the condition $\frac{d\left(A(t)\,\Phi_A(t)\right)}{A(t)} = \frac{d\left(L(t)\,\Phi_L(t)\right)}{L(t)}$[30] which has the interpretation of matching the standardized change in "Dollar Modified Duration", that is Modified Duration times price, across assets and liabilities. Now, for a portfolio at time t with value $B(t)$ and Basis Risk measure $\Phi(t)$ the transition $\frac{d\left(B(t)\,\Phi(t)\right)}{B(t)}$ expressed relative to the transition of the basis $x(t)$ is[31]

$$\frac{d\left(B(t)\,\Phi(t)\right)}{B(t)} = \left(\frac{\sum_{i=1}^N C_i\,P(t,T_i)\,G(\omega_t,t,T_i)}{\sum_{i=1}^N C_i\,P(t,T_i)}\right) dt + \left(\frac{\sum_{i=1}^N C_i\,P(t,T_i)\,H(\omega_t,t,T_i)}{\sum_{i=1}^N C_i\,P(t,T_i)}\right) dx(t) \tag{17}$$

where $\quad H(\omega_t,t,T) = \eta(\omega_t,t,T) + \xi(\omega_t,t,T)^2$

$$G(\omega_t,t,T) = \alpha(\omega_t,t,T) + \psi(\omega_t,t)^2\,\xi(\omega_t,t,T)\,\eta(\omega_t,t,T) + \xi(\omega_t,t,T)\,r(t) - m(\omega_t,t)\,H(\omega_t,t,T).$$

The terms $H(\omega_t,t,T)$ and $G(\omega_t,t,T)$ are the coefficients of $dx(t)$ and dt respectively for the process $\frac{d\left(P(t,T)\,\xi(\omega_t,t,T)\right)}{P(t,T)}$. Consequently to ensure the evolution of the asset portfolio equals that of the liability

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[29] Suppose we model the dynamics of the forward rate volatility structure in the form of a stochastic differential equation

$$d\gamma(\omega_t,t,T) = \left(\zeta(\omega_t,t,T) + \lambda(\omega_t,t)\,\delta(\omega_t,t,T)\right)dt + \delta(\omega_t,t,T)\,dW(t)$$

where $\zeta(\omega_t,t,T)$ is the drift of the forward rate volatility structure in the equivalent risk neutral economy, $\delta(\omega_t,t,T)$ is the volatility of the forward rate volatility structure, and $W(t)$ is the same Brownian motion that introduces uncertainty into the bond market. Further suppose we choose as our basis the Brownian motion $W(t)$ itself. In this case $\xi(\omega_t,t,T) = \int_t^T \gamma(\omega_t,t,v)\,dv$ and the dynamics of $\xi(\omega_t,t,T)$ can be expressed as

$$d\xi(\omega_t,t,T) = \left(\int_t^T \zeta(\omega_t,t,v) + \lambda(\omega_t,t)\,\delta(\omega_t,t,v)\,dv - \gamma(\omega_t,t,t)\right)dt + \left(\int_t^T \delta(\omega_t,t,v)\,dv\right)dW(t).$$

[30] This is because the Duration immunization conditions ensure $A(t) = L(t)$, $\Phi_A(t) = \Phi_L(t)$, and $A(t+dt) = L(t+dt)$. The reason for considering the condition $\frac{d\left(A(t)\,\Phi_A(t)\right)}{A(t)} = \frac{d\left(L(t)\,\Phi_L(t)\right)}{L(t)}$ instead of $d\Phi_A(t) = d\Phi_L(t)$ is purely for the convenience of later expressions and interpretations.

[31] See Appendix 1.

portfolio over two periods in time the following two conditions must hold in addition to the Duration immunization conditions (9) and (10):

$$\frac{\sum_{i=1}^{N_A} C_i^A\, P(t,T_i^A)\, H(\omega_t,t,T_i^A)}{\sum_{i=1}^{N_A} C_i^A\, P(t,T_i^A)} = \frac{\sum_{i=1}^{N_L} C_i^L\, P(t,T_i^L)\, H(\omega_t,t,T_i^L)}{\sum_{i=1}^{N_L} C_i^L\, P(t,T_i^L)} \tag{18}$$

$$\frac{\sum_{i=1}^{N_A} C_i^A\, P(t,T_i^A)\, G(\omega_t,t,T_i^A)}{\sum_{i=1}^{N_A} C_i^A\, P(t,T_i^A)} = \frac{\sum_{i=1}^{N_L} C_i^L\, P(t,T_i^L)\, G(\omega_t,t,T_i^L)}{\sum_{i=1}^{N_L} C_i^L\, P(t,T_i^L)} \tag{19}$$

Since the above conditions result from the intended spirit of Convexity within an immunization context, that is match the Convexity of assets to that of liabilities to reduce the frequency of re-immunization, we propose the following definition for Convexity:

Definition (CONVEXITY):

Take as given at time t the term structure $P(t,T)$ and the computed coefficients $G(\omega_t,t,T)$ and $H(\omega_t,t,T)$ of the process $\frac{d\left(P(t,T)\,\xi(\omega_t,t,T)\right)}{P(t,T)} = G(\omega_t,t,T)\,dt + H(\omega_t,t,T)\,dx(t)$ where $x(t)$ is the basis and $\xi(\omega_t,t,T)$ is the Basis Risk of the pure discount bond at time t with maturity date T. The *Convexity* of a portfolio containing positive default-free cash flows C_i to occur at respective times T_i, where $T_i > t$, is two dimensional with components

$$\begin{aligned} Convexity_1 &= \frac{\sum_{i=1}^{N} C_i\, P(t,T_i)\, H(\omega_t,t,T_i)}{\sum_{i=1}^{N} C_i\, P(t,T_i)} \\ Convexity_2 &= \frac{\sum_{i=1}^{N} C_i\, P(t,T_i)\, G(\omega_t,t,T_i)}{\sum_{i=1}^{N} C_i\, P(t,T_i)} \end{aligned} \tag{20}$$

From the above definition we can interpret $H(\omega_t,t,T)$ and $G(\omega_t,t,T)$ as Convexity_1 and Convexity_2 at time t for a single cash flow to occur at time T. Consequently the above Convexity measure of a portfolio has the property that it is the weighted average Convexity of each asset in the portfolio where the weighting scheme is the usual present value weighting scheme.

An interpretation for Convexity as defined above is that the change in Dollar Modified Duration can be random and it has an exposure to changes in the basis measured by Convexity_1 and an additional expected change component measured by Convexity_2. An obvious observation at this point is that the above definition of Convexity is two-dimensional. This appears unusual given that the traditional measure for Convexity, namely $\frac{\partial^2 B}{\partial r^2} \Big/ B$, is not. This issue is the subject of the following subsections where they are reconciled. At this point however, if we have to choose between the two components of Convexity then the one which is of higher order importance is Convexity_1.[32]

A: Consistency with Traditional Convexity

Traditional measures of Convexity are obtained by computing $\frac{\partial^2 B}{\partial r^2} \Big/ B$ in the setting of either Fisher-Weil where the price at time *t* of a default-free portfolio of positive cash flows is computed via $B = \sum_{i=1}^{N} C_i \exp\left\{-\int_t^T r + \delta(v)\,dv\right\}$, or Macaulay which is the special case of Fisher-Weil when $\delta(v) = 0$ for all *v*. For both cases the resulting Convexity measure for this portfolio is

$$\frac{\partial^2 B}{\partial r^2} \Big/ B = \frac{\sum_{i=1}^{N} (T_i - t)^2 C_i P(t,T_i)}{\sum_{i=1}^{N} C_i P(t,T_i)}$$

where $P(t,T) = \exp\left\{\int_t^T r + \delta(v)\,dv\right\}$ which is the term structure of pure discount bonds at time *t*. Consequently traditional Convexity can be interpreted as the present value weighted average of the time to each cash flow squared.

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[32] From (17) we can see that Convexity_1 is the coefficient of $dx(t)$ which is of order $\sqrt{dt}$ whereas Convexity_2 is the coefficient of dt. Hence Convexity_1 is of higher order importance when matching Convexity.

To reconcile the above traditional Convexity measure with the Convexity definition (20) first recall from Section III-B that the only class of volatility structures consistent with the Fisher-Weil Duration measure is $\gamma(\omega_t,t,T) = \sigma(\omega_t,t)$. Second, a basis must be chosen and to simplify calculations set $m(\omega_t,t) = 0$ and $\psi(\omega_t,t) = \sigma(\omega_t,t)$. This can be interpreted as choosing the short rate as the basis with an adjustment so that expected changes in the short rate are always zero in the equivalent risk-neutral economy. With this choice the resulting Basis Risk measure of a single default-free cash flow is $\xi(\omega_t,t,T) = T-t$ which has intertemporal dynamics described by (16) when $\alpha(\omega_t,t,T) = -1$ and $\eta(\omega_t,t,T) = 0$. Consequently the functions $H(\omega_t,t,T)$ and $G(\omega_t,t,T)$ in (17) simplify to $(T-t)^2$ and $\left((T-t)r(t) - 1\right)$ respectively resulting in a Convexity measure of

$$Convexity_1 = \frac{\sum_{i=1}^{N} (T_i-t)^2 \, C_i \, P(t,T_i)}{\sum_{i=1}^{N} C_i \, P(t,T_i)}$$

$$Convexity_2 = r(t) \frac{\sum_{i=1}^{N} (T_i-t) \, C_i \, P(t,T_i)}{\sum_{i=1}^{N} C_i \, P(t,T_i)} - 1 = r(t) \text{ x } Duration - 1$$

Convexity_1 above is identical to the traditional Convexity measure and at each fixed point in time Convexity_2 is a linear function of Duration. Consequently if we wish to engage in an immunization strategy where present value, Duration, and Convexity are matched across assets and liabilities then we can ignore Convexity_2 because it is redundant. In this sense the only relevant component of Convexity in the traditional settings of Fisher-Weil or Macaulay is indeed the traditional measure of $\frac{\partial^2 B}{\partial r^2} \Big/ B$.

B: Convexity in the CIR (1979) framework

Computing the Convexity measure $\frac{\partial^2 B}{\partial r^2} \Big/ B$ can be done in the framework of CIR (1979) since the value at time *t* of a portfolio containing default-free cash flows takes the functional form $B(r(t),t)$ where $r(t)$ is the short term interest rate at time *t*. Is this measure Convexity in the CIR (1979) framework also consistent with the definition of Convexity provided in this paper?

To understand the relationship between these two concepts of Convexity first choose the short rate as the basis for computing Convexity as defined in this paper. This choice is motivated from the observation that Convexity in the CIR (1979) setting considers changes in a portfolio's value with respect to movements in the short rate. Now consider the dynamics of $\left(P(t,T)\,\xi(\omega_t,t,T)\right)$ with respect to this basis within the Markovian spot rate framework of CIR(1979). Letting $P(r(t),t,T)$ denote the functional form for the price at time t of a bond that matures at time T, the dynamics of $\left(P(t,T)\,\xi(\omega_t,t,T)\right)$ can be written in the form[33]

$$\frac{d\left(P(t,T)\,\xi(\omega_t,t,T)\right)}{P(t,T)} = G(\omega_t,t,T)\,dt + H(\omega_t,t,T)\,dr(t) \tag{21}$$

where
$$H(\omega_t,t,T) = \frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2} \Big/ P(r(t),t,T)$$

$$G(\omega_t,t,T) = 1 + \left(r(t) - \frac{\partial\,\theta(r(t),t)}{\partial r(t)} \right) \xi(\omega_t,t,T) - \left(\theta(r(t),t) + \sigma(r(t),t)\frac{\partial \sigma(r(t),t)}{\partial r(t)} \right) H(\omega_t,t,T).$$

Imposing this structure onto the definition of Convexity in (20) for a portfolio whose value is computed via $B(r(t),t) = \sum_{i=1}^{N} C_i\, P(r(t),t,T_i)$ yields

$$Convexity_1 = \frac{\sum_{i=1}^{N} C_i\, P(t,T_i)\left(\frac{\partial^2 P(r(t),t,T_i)}{\partial r(t)^2} \Big/ P(r(t),t,T_i) \right)}{\sum_{i=1}^{N} C_i\, P(t,T_i)} = \frac{\partial^2 B(r(t),t)}{\partial r(t)^2} \Big/ B(r(t),t)$$

$$Convexity_2 = 1 + \left(r(t) - \frac{\partial\,\theta(r(t),t)}{\partial\, r(t)} \right) \text{x} \begin{pmatrix} Modified \\ Duration \end{pmatrix}$$
$$- \left(\theta(r(t),t) + \sigma(r(t),t)\,\frac{\partial\,\sigma(r(t),t)}{\partial\, r(t)} \right) \text{x}\; Convexity_1$$

[33] See Appendix 2.

The above shows that Convexity$_2$ is a linear function of Modified Duration and Convexity$_1$. Once again we see that if we wish to engage in an immunization strategy where present value, Duration, and Convexity are matched across assets and liabilities then matching Convexity$_2$ is redundant. Consequently a relevant measure of Convexity in the CIR (1979) setting is simply Convexity$_1$ which is exactly the traditional measure of Convexity.

An intuitive interpretation for why Convexity as defined in this paper is two dimensional whereas it is only one dimensional in the CIR (1979) framework can be obtained from thinking of a liability portfolio's value evolving on a two-period binomial tree. Now consider immunizing this liability portfolio with an asset portfolio. Over one period we require two conditions to be met by this asset portfolio to ensure that its value imitates the liability portfolio's outcomes. Duration immunization provides these two conditions; the present value matching condition and the Basis Risk matching condition. However over two period there are four possible paths that the liability portfolio's value can take. Consequently we require four conditions on the asset portfolio to ensure it has the same value as the liability portfolio in these four states of nature. The two additional conditions correspond to the two Convexity conditions provided in this paper. However in a path-independent term structure setting such as that of CIR(1979) the up/down and down/up movements of the liability portfolio are perceived to be identical. Consequently one of the Convexity conditions is redundant.

C: A Scheme for Calculating Convexity in Practice

Here we return to the practical scheme for computing Duration presented in Section III-A and extend it to include Convexity. As a reminder the chosen basis is obtained by setting $m(\omega_t,t) = 0$ and $\psi(\omega_t,t) = 1$ which allows us to interpret bond volatility as Basis Risk. Further it was argued that a reasonable local approximation to the bond volatility structure $\Gamma(\omega_t,t,T)$ is obtained by restricting it to be purely a function of time-to-maturity $\Gamma(\tau)$ and estimating it using recent historical data via $\Gamma(\tau) = \sqrt{\frac{1}{\Delta t}\mathrm{VAR}\left[\Delta\ln(P(t,t+\tau))\right]}$. The resulting Basis Risk measure of a default-free portfolio containing positive cash flows is

$$\begin{pmatrix} Basis \\ Risk \end{pmatrix} = \frac{\sum_{i=1}^{N} \Gamma(T_i - t)\, C_i\, P(t,T_i)}{\sum_{i=1}^{N} C_i\, P(t,T_i)}.$$

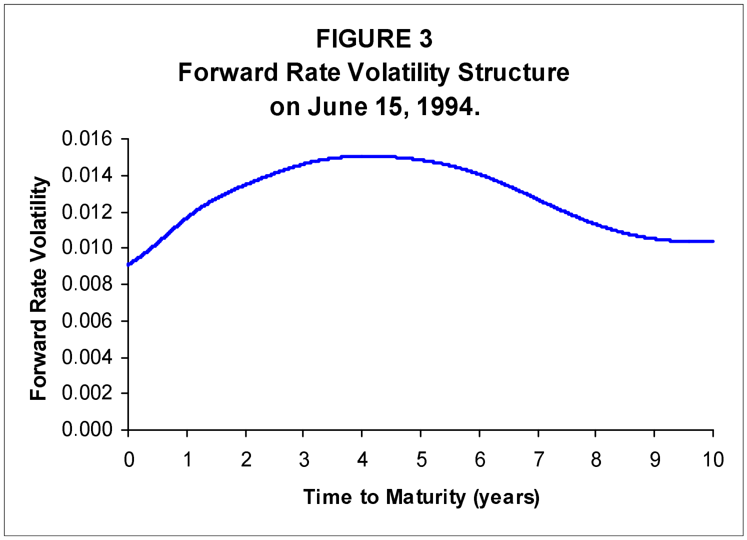
To compute Convexity we need to characterize the evolution of Basis Risk for single cash flows as in (16). Here $\xi(\omega_t,t,T) = \Gamma(T-t)$ which implies $\alpha(\omega_t,t,T) = -\frac{\partial\Gamma(\tau)}{\partial\tau}$ and $\eta(\omega_t,t,T) = 0$ so the resulting Convexity measure is

$$Convexity_1 = \frac{\sum_{i=1}^{N} \Gamma(T_i - t)^2\, C_i\, P(t,T_i)}{\sum_{i=1}^{N} C_i\, P(t,T_i)}$$

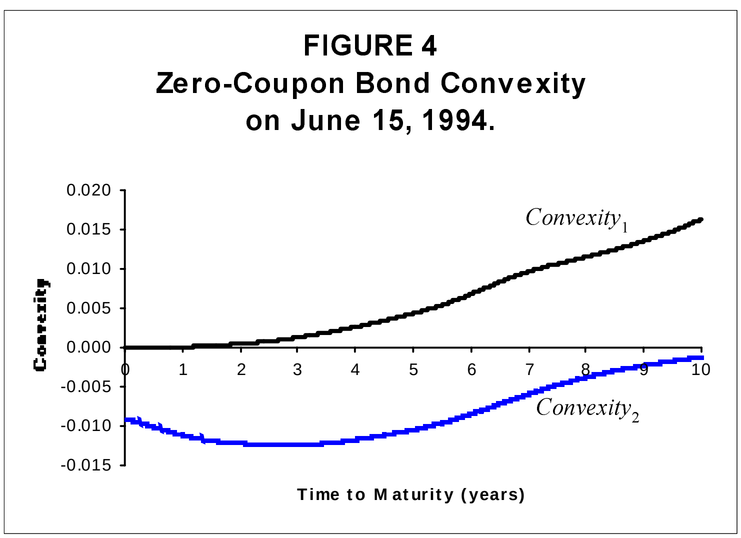
$$Convexity_2 = \frac{\sum_{i=1}^{N} -\frac{\partial\Gamma(T_i - t)}{\partial(T_i - t)}\, C_i\, P(t,T_i)}{\sum_{i=1}^{N} C_i\, P(t,T_i)} + r(t) \text{ x} \begin{pmatrix} Basis \\ Risk \end{pmatrix}.$$

As an aside, observe that when creating an immunization strategy where present values, Durations and Convexities are matched across assets and liabilities we can re-center Convexity$_2$ by subtracting $r(t)$ x (*Basis Risk*) without affecting the strategy. This is because Duration immunization already requires the matching of Basis Risks. We can interpret this re-centered Convexity as the weighted average of forward rate volatilities since $\frac{\partial\Gamma(\tau)}{\partial\tau}$ is the forward rate volatility structure.

To demonstrate the calculation of Convexity we continue the numerical example presented in Section III-A, namely consideration of a 9 year, \$100 face value bond with a coupon rate of 5% per annum payable semi-annually on June 15, 1994. Estimates for the yield curve and $\Gamma(\tau)$ on June 15, 1994 appear in Section III-A. The computed bond value is \$86.2937 and the bond's Basis Risk is 0.092777. The only remaining item required for calculating Convexity is an estimate for $\frac{\partial\Gamma(\tau)}{\partial\tau}$. This is obtained by numerically differentiating the volatility structure estimate of $\Gamma(\tau)$ which is depicted in Figure 3 below.[34]



We now have sufficient information to compute the Convexity of any single cash flow with a time-to-maturity of τ; denote the Convexity_1 and Convexity_2 for this single cash flow as $H(\tau)$ and $G(\tau)$ respectively. In particular $H(\tau) = \Gamma(\tau)^2$ and $G(\tau) = -\frac{\partial \Gamma(\tau)}{\partial (\tau)} + r\,\Gamma(\tau)$ where r is the short rate from the observed yield curve. Both components of the Convexity measure for all single cash flows is depicted in Figure 4.



[34] Note that traditional Convexity in the Fisher-Weil or Macaulay setting implies that the forward rate volatility structure should be constant across maturity (see Section IV-A). This does not appear to be the case in Figure 3. Whether this deviation is significant in an immunization context remains an empirical question.

The Convexity of the coupon bond in question can be obtained by observing that the Convexity of a portfolio is the present value weighted average of the Convexities from each cash flow in the portfolio, that is

$$Convexity_1 = \frac{\sum_{i=1}^{N} H(T_i - t)\, C_i\, P(t,T_i)}{\sum_{i=1}^{N} C_i\, P(t,T_i)} = 0.01000282$$

$$Convexity_2 = \frac{\sum_{i=1}^{N} G(T_i - t)\, C_i\, P(t,T_i)}{\sum_{i=1}^{N} C_i\, P(t,T_i)} = 0.00804985\,.$$

The above information can be used to find a portfolio that matches the present value, Duration and Convexity of the coupon bond. This requires two cash flows in the portfolio where maturities and quantities of these cash flows must be chosen. That is, we want to solve for a, b, T_1 and T_2 in the following system of equations

$$\begin{aligned}
a\,P(t,T_1) &+ b\,P(t,T_2) = 86.2937 \\
\frac{a\,P(t,T_1)}{86.2937}\,\Gamma(T_1 - t) &+ \frac{b\,P(t,T_2)}{86.2937}\,\Gamma(T_2 - t) = 0.092777 \\
\frac{a\,P(t,T_1)}{86.2937}\,H(T_1 - t) &+ \frac{b\,P(t,T_2)}{86.2937}\,H(T_2 - t) = 0.01000282 \\
\frac{a\,P(t,T_1)}{86.2937}\,G(T_1 - t) &+ \frac{b\,P(t,T_2)}{86.2937}\,G(T_2 - t) = -0.00804985\,.
\end{aligned}$$

A solution to the above system is $a = 26.4516$, $b = 119.0285$, $T_1 = 2.62$ and $T_1 = 8.86$. This has the interpretation that a portfolio with \$26.4516 invested in a 2.62 year zero-coupon bond and \$119.0285 invested in an 8.86 year zero-coupon bond behaves like the original coupon bond over a $2dt$ time period.

V. DURATION, CONVEXITY AND HIGHER ORDER HEDGING IN A MULTI-FACTOR TERM STRUCTURE SETTING

This section will appear in the next version of the paper.

VI. SUMMARY AND CONCLUSIONS

This paper generalizes the traditional measures of Duration and Convexity in a manner that circumvents the need to think of them in terms of partial derivatives. This is necessary within the context of modern-day term structure models where the term structure at a single point in time generally cannot be summarized by a finite number of state variables.

Here Duration can be thought of as "the time-to-maturity of the zero coupon bond that *behaves like* the portfolio of cash flows under consideration". The term "behaves like" is meant in the sense that the zero coupon bond and the portfolio have the same current value and will have identical realizations over the next instant in time. The matching of realizations is obtained by matching the volatility of the zero coupon bond to that of the portfolio hand hence volatility is closely related to the calculation of Duration. In a multi-factor term structure setting a single zero-coupon bond cannot achieve this goal however a portfolio containing N zeros can where N is the number of Brownian motions introducing uncertainty into the bond market. Consequently Duration in a multi-factor setting becomes an N-dimensional vector.

Convexity can be thought of as measure of how to maintain a duration based hedge over a longer period of time. The interesting result of this measure is that it is generally two-dimensional. This results because the term structure is allowed to be generically path-dependent. The first component of Convexity is related to a combination of the term structure's volatility and the volatility of this volatility whereas the second component is related to expected changes in the term structure's volatility. However under the traditional circumstances of the Fisher-Weil setting and the path-independent setting such as CIR (1979) then the traditional one-dimension convexity measure is obtained. In particular the second component of the Convexity measure derived in this paper becomes redundant.

In order to calculate the Duration and Convexity measures as defined in this paper requires the term structure, the volatility of the term structure, and a description of how this volatility changes over time; the latter is required only for the Convexity measure. This paper has been nondescript in terms of these components so the specifications can be left to the user. However the case where the term structure's volatility is purely a function of time-to-maturity is considered as an example.

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APPENDIX 1

Given the dynamics for $P(t,T)$ in the case of one Brownian motion and $\xi(\omega_t,t,T)$ in equations (1) and (15) respectively, the dynamics for the process $\big(P(t,T)\,\xi(\omega_t,t,T)\big)$ can be expressed as

$$d\big(P(t,T)\,\xi(\omega_t,t,T)\big) = P(t,T)\;d\xi(\omega_t,t,T) + \xi(\omega_t,t,T)\;dP(t,T) + d\big\langle P(t,T)\,,\,\xi(\omega_t,t,T)\big\rangle_t$$

that is
$$\begin{aligned}\frac{d\big(P(t,T)\,\xi(\omega_t,t,T)\big)}{P(t,T)} &= \big(\alpha(\omega_t,t,T) + \lambda(\omega_t,t)\,\eta(\omega_t,t,T)\,\psi(\omega_t,t)\big)dt + \eta(\omega_t,t,T)\,\psi(\omega_t,t)\,dW(t)\\ &+ \xi(\omega_t,t,T)\big(r(t) + \lambda(\omega_t,t)\,\Gamma(\omega_t,t,T)\big)\,dt + \xi(\omega_t,t,T)\,\Gamma(\omega_t,t,T)\,dW(t)\\ &+ \Gamma(\omega_t,t,T)\,\eta(\omega_t,t,T)\,\psi(\omega_t,t)\;dt\;.\end{aligned}$$

Observing that $\Gamma(\omega_t,t,T) = \xi(\omega_t,t,T)\,\psi(\omega_t,t)$ and simplifying yields

$$\begin{aligned}\frac{d\big(P(t,T)\,\xi(\omega_t,t,T)\big)}{P(t,T)} &= \Big(\alpha(\omega_t,t,T) + \xi(\omega_t,t,T)\,r(t) + \psi(\omega_t,t)^2\,\xi(\omega_t,t,T)\,\eta(\omega_t,t,T)\\ &+ \lambda(\omega_t,t)\,\psi(\omega_t,t)\big(\eta(\omega_t,t,T) + \xi(\omega_t,t,T)^2\big)\Big)\,dt\\ &+ \psi(\omega_t,t)\big(\eta(\omega_t,t,T) + \xi(\omega_t,t,T)^2\big)\,dW(t)\;.\end{aligned}$$

Expressing the evolution of the process $\big(P(t,T)\,\xi(\omega_t,t,T)\big)$ in terms of the basis factor $x(t)$ yields

$$\frac{d\big(P(t,T)\,\Gamma(\omega_t,t,T)\big)}{P(t,T)} = G(\omega_t,t,T)\,dt + H(\omega_t,t,T)\,dx(t)$$

where
$$\begin{aligned}H(\omega_t,t,T) &= \eta(\omega_t,t,T) + \xi(\omega_t,t,T)^2\\ G(\omega_t,t,T) &= \alpha(\omega_t,t,T) + \psi(\omega_t,t)^2\,\xi(\omega_t,t,T)\,\eta(\omega_t,t,T) + \xi(\omega_t,t,T)\,r(t) - m(\omega_t,t)\,H(\omega_t,t,T)\end{aligned}$$

Now observing from (6) that $B(t)\,\Phi(t) = \sum_{i=1}^{N} C_i\,P(t,T_i)\,\xi(\omega_t,t,T_i)$ we can write

$$\begin{aligned}\frac{d\big(B(t)\,\Phi(t)\big)}{B(t)} &= \frac{\sum_{i=1}^{N} C_i\,d\big(P(t,T_i)\,\xi(\omega_t,t,T_i)\big)}{\sum_{i=1}^{N} C_i\,P(t,T_i)}\\ &= \left(\frac{\sum_{i=1}^{N} C_i\,P(t,T_i)\,G(\omega_t,t,T_i)}{\sum_{i=1}^{N} C_i\,P(t,T_i)}\right)dt + \left(\frac{\sum_{i=1}^{N} C_i\,P(t,T_i)\,H(\omega_t,t,T_i)}{\sum_{i=1}^{N} C_i\,P(t,T_i)}\right)dx(t)\end{aligned}$$

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APPENDIX 2

The dynamics for pure discount bonds takes the form

$$\frac{dP(t,T)}{P(t,T)} = \Big(r(t) + \lambda(\omega_t,t)\,\Gamma(\omega_t,t,T)\Big)\,dt + \Gamma(\omega_t,t,T)\,dW(t). \tag{A-1}$$

In the Markovian framework of CIR(1979) the functional form for the prices of all pure discount bonds can be written as $P(r(t),t,T)$ and the dynamics of the short rate can be expressed as

$$dr(t) = \Big(\theta(r(t),t) + \lambda(\omega_t,t)\,\sigma(r(t),t)\Big)\,dt + \sigma(r(t),t)\,dW(t).$$

Applying Itô's lemma to $P(r(t),t,T)$ shows that the dynamics of bond prices can also be expressed as

$$dP(r(t),t,T) = \left(\frac{\partial P(r(t),t,T)}{\partial r(t)}\mu(\omega_t,t) + \frac{\partial P(r(t),t,T)}{\partial t} + \frac{1}{2}\frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2}\sigma(r(t),t)^2\right)dt + \left(\frac{\partial P(r(t),t,T)}{\partial r(t)}\sigma(r(t),t)\right)dW(t) \tag{A-2}$$

where $\mu(\omega_t,t) = \theta(r(t),t) + \lambda(\omega_t,t)\,\sigma(r(t),t)$. Comparing the drift and diffusion coefficients across (A-1) and (A-2) provides two restrictions implied by the addition of the Markovian framework, namely

$$P(t,T)\,\Gamma(\omega_t,t,T) = \frac{\partial P(r(t),t,T)}{\partial r(t)}\sigma(r(t),t) \tag{A-3}$$

$$P(t,T)\Big(r(t) + \lambda(\omega_t,t)\,\Gamma(\omega_t,t,T)\Big) = \frac{\partial P(r(t),t,T)}{\partial r(t)}\mu(\omega_t,t) + \frac{\partial P(r(t),t,T)}{\partial t} + \frac{1}{2}\frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2}\sigma(r(t),t)^2. \tag{A-4}$$

Observing from (5) that $\xi(\omega_t,t,T) = \frac{\Gamma(\omega_t,t,T)}{\sigma(r(t),t)}$ when the short rate is the chosen basis, from (A-2) we have $P(t,T)\,\xi(\omega_t,t,T) = \frac{\partial P(r(t),t,T)}{\partial r(t)}$. Consequently the dynamics for $\Big(P(t,T)\,\Gamma(\omega_t,t,T)\Big)$ can be expressed as

$$\begin{aligned} d\Big(P(t,T)\,\xi(\omega_t,t,T)\Big) &= d\left(\frac{\partial P(r(t),t,T)}{\partial r(t)}\right) \\ &= \left(\frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2}\Big(\theta(r(t),t) + \lambda(\omega_t,t)\,\sigma(r(t),t)\Big) + \frac{\partial^2 P(r(t),t,T)}{\partial r(t)\,\partial t} + \frac{1}{2}\sigma(r(t),t)^2\frac{\partial^3 P(r(t),t,T)}{\partial r(t)^3}\right)dt \\ &\quad + \left(\frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2}\sigma(r(t),t)\right)dW(t) \end{aligned} \tag{A-5}$$

However, from (A-3) and (A-4)

$$\frac{\partial^2 P(r(t),t,T)}{\partial r(t)\,\partial t} = \frac{\partial}{\partial r(t)}\left(r(t)\,P(r(t),t,T) - \frac{\partial P(r(t),t,T)}{\partial r(t)}\theta(r(t),t) - \frac{1}{2}\frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2}\sigma(r(t),t)^2 \right)$$

and rearranging provides

$$\frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2}\theta(r(t),t) + \frac{\partial^2 P(r(t),t,T)}{\partial r(t)\,\partial t} + \frac{1}{2}\frac{\partial^3 P(r(t),t,T)}{\partial r(t)^3}\sigma(r(t),t)^2$$
$$= P(r(t),t,T) + \left(r(t) - \frac{\partial\theta(r(t),t)}{\partial r(t)} \right)\frac{\partial P(r(t),t,T)}{\partial r(t)} - \sigma(r(t),t)\frac{\partial\sigma(r(t),t)}{\partial r(t)}\frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2} \;.$$

Substituting the above into (A-5) provides the expression

$$d\Big(P(t,T)\,\xi(\omega_t,t,T)\Big) = \left(P(r(t),t,T) + \Big(r(t) - \tfrac{\partial\theta(r(t),t)}{\partial r(t)}\Big)\frac{\partial P(r(t),t,T)}{\partial r(t)} + \Big(\lambda(\omega_t,t) - \tfrac{\partial\sigma(r(t),t)}{\partial r(t)}\Big)\frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2}\sigma(r(t),t) \right) dt$$
$$+ \left(\frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2}\sigma(r(t),t) \right) dW(t)$$

Expressing this dynamic in terms of the spot rate transitions, since the spot rate is the chosen basis, and again observing that $\xi(\omega_t,t,T) = \frac{\partial P(r(t),t,T)}{\partial r(t)} \Big/ P(r(t),t,T)$ yields

$$\frac{d\Big(P(t,T)\,\xi(\omega_t,t,T)\Big)}{P(t,T)} = G(\omega_t,t,T)\,dt + H(\omega_t,t,T)\,dr(t)$$

where $$H(\omega_t,t,T) = \frac{\partial^2 P(r(t),t,T)}{\partial r(t)^2} \Big/ P(r(t),t,T)$$

$$G(\omega_t,t,T) = 1 + \left(r(t) - \frac{\partial\theta(r(t),t)}{\partial r(t)} \right)\xi(\omega_t,t,T) - \left(\theta(r(t),t) + \sigma(r(t),t)\frac{\partial\sigma(r(t),t)}{\partial r(t)} \right) H(\omega_t,t,T) \;.$$

■