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REAL INVESTMENTS UNDER KNIGHTIAN UNCERTAINTY

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Abstract

In a model of real investments with Knightian uncertainty, decision makers deviate from expected utility theory by showing excessive risk aversion and focusing on no regret moves. Within the model, a positive net present value is no longer sufficient to ensure that a real investment is undertaken. Furthermore, the value of being able to hedge increases drastically.

The model could explain deviations from the net present value rule in industries where Knightian uncertainty is high. For example, high hurdle rates for venture capital, and stalled investments in several broadband markets are consistent with the model.

Introduction

The standard net present value (NPV) rule for whether a firm should take on a real investment is one of the most important and widely used rules in financial decision making. It was originally introduced by Irving Fisher (1930) and is beautiful in its simplicity. In the absence of capital constraints, it reads: *Find an appropriate discount factor for the project and calculate the expectation of the sum of discounted cash flows. If the NPV is greater than zero, invest in the project, otherwise do not invest.* In an economy in which the marginal investing is well diversified, the correct discount factor will include systematic but not idiosyncratic risk. By far the most common method for deciding the discount factor is to use the CAPM, see e.g. Copeland, Koller, and Murrin (1994) and Ibbotson Associates (2003). [1]

Although the NPV rule has gained usage with the growth of MBA education over the last decades, it is by no means the only technique used. Graham and Harvey (2001) report that over 50% of the respondents in their survey sample (CFOs of mid-size and large companies) use the payback method always or often in capital budgeting decisions, compared with 75% that use the net present value method. In a similar survey for Sweden's 500 largest companies, Sandahl and Sjögren (2003) show that almost 35% of the respondents do *not* use NPV based methods for capital budgeting. Although the fraction of companies using NPV analyses is steadily rising[2], there is still a significant fraction of companies that use capital budgeting analyses that are not NPV based. The companies in the surveys are not small (the smallest company in the survey has revenues of about $100 million in Graham and Harvey (2001) and $50 million in Sandahl and Sjögren (2003)) and considering the extent to which the NPV rule has been taught to MBA students over the last decades one might expect that the numbers would be higher.

In other situations, the NPV rule is used, but in unorthodox ways. Venture capital is one such example.[3] It is common practice to require expected rates of returns of 50% or higher to take on a project. Plummer (1987) reports discount rates of 50%-70% for startups, and 35%-50%

for late stage financing[4]. To motivate such high hurdle rates, one would need extremely high betas. As the (large) risks involved in venture capital are often idiosyncratic, this seems to be a violation of the standard NPV rule.[5]

Another area where hurdle rates seem to be significantly higher than motivated by the NPV rule is catastrophe insurance (Froot, Murphy, Stern, and Usher 1995, Froot and O'Connell 1997). Moreover, within capital budgeting, firms generally seem to use higher hurdle rates than cost of capital, as noted by Poterba and Summers (1995).

What is special with venture capital? Venture capital is risky in the sense that the spread of the probability distribution for payoffs can be very high. Another profound feature, which we will stress, is the great difficulty to even *assess* the probability distribution for different payoffs. This is so called *Knightian uncertainty*, as opposed to *risk* by which we mean spread of payoffs with known probability distribution. Knightian uncertainty is neither taken into account in classical von-Neumann Morgenstern expected utility theory, nor in the standard NPV rule (In the standard NPV rule, an "objective" expectations operator is used, and risk aversion is taken into account by using a higher discount rate). However, in real life decision making it is often important (See for example the article in Harvard Business Review by Courtney, Kirkland, and Vignerie (1997). They discuss different types of uncertainty in strategic decision making and distinguish between four types of uncertainty in investment decisions. The uncertainty in the latter two are closely related to the concept of Knightian uncertainty).

This paper introduces a model with investors who are averse towards Knightian uncertainty, to explain why a positive NPV is not a sufficient condition to ensure that an investment is undertaken. Models with Knightian uncertainty lie on robust theoretical grounds: like von-Neumann Morgenstern theory, they are based on decision theoretical axioms. Thus, if one is willing to accept the axioms, aversion towards Knightian uncertainty is rational. We will argue that it is natural to use this concept for real investments. Moreover, we will see that excessive risk aversion compared with the standard NPV rule arises naturally within a framework with Knightian uncertainty.

To see how Knightian uncertainty influences decision making, let us review a variation of the classical Ellsberg paradox (Ellsberg 1961):

Example 1 : Ellsberg three-color example

An urn contains 90 balls. These are either red, yellow or blue. A decision maker knows that 30 balls are red, but not the number of blue or yellow balls. He is first given the choice between the following "Red" and "Blue" lotteries:

1. RL: Pick a ball from the urn. If it is red you earn $10, otherwise nothing.
2. BL: Pick a ball from the urn. If it is blue you earn $10, otherwise nothing.

In laboratory tests, people tend to choose RL over BL:

$$\mathrm{RL} \succ \mathrm{BL}.^{6} \tag{1}$$

Next, the decision maker chooses between the following "Not Red" and "Not Blue" lotteries:

1. NRL: Pick a ball from the urn. If it is not red you earn $10, otherwise nothing.
2. NBL: Pick a ball from the urn. If it is not blue you earn $10, otherwise nothing.

In laboratory tests, people tend to choose NRL over NBL:

$$\mathrm{NRL} \succ \mathrm{NBL}. \tag{2}$$

Together, these two preferences (1,2) are not consistent with subjective expected utility (SEU) optimization, as an expected utility maximizing decision maker should assume (subjective) probabilities for the likeliness of the three events $\mathbb{P}(\text{red}) = \alpha$, $\mathbb{P}(\text{blue}) = \beta$, $\mathbb{P}(\text{yellow}) = 1 - \alpha - \beta$ (where $\alpha = 1/3$ if the decision maker is rational). However, (1) would imply that

$$\alpha > \beta, \tag{3}$$

whereas (2) on the other hand would imply that

$$1 - \alpha > 1 - \beta, \tag{4}$$

so we have a contradiction. Thus, a decision maker expressing these preferences is not an SEU optimizer.

□

Subjective expected utility theory breaks down in this example, because many people prefer known probabilities over unknown. The preferred RL and NRL lotteries with known probabilities are so called *roulette lotteries* whereas the BL and NBL lotteries are *horse lotteries.* Now, in many – probably even in a majority of – investment decisions, one can argue that *the* major difficulty is to assess the probabilities for different outcomes. For example, investments in R&D, in new products for which demand is unknown, in geographically new markets and after unique shocks, will typically have significant components of Knightian uncertainty. If decision makers are averse to Knightian uncertainty this should have an effect in markets where such uncertainty is high. We will see that the effect is indeed important: A decision maker facing Knightian uncertainty will

1. *Act as if cost of capital has increased.* Specifically, he will:

    - Decrease the amount invested in uncertain projects when choosing how much to invest in an uncertain project and how much to invest in a riskfree project.
    - Undertake fewer projects when choosing between investing in an uncertain project or in a riskfree project.
    - Require a higher internal rate of return to undertake a project when choosing between investing in an uncertain project or in a riskfree project.

2. *Need to supplement the NPV rule with other value measures.* Specifically,

    - As uncertainty increases, a larger fraction of projects that are "recommended" by the NPV rule will be unwanted by the decision maker.

3. *Invest in different types of projects than a (more) risk averse decision maker would.* Specifically, he will:

    - Behave as if he is extremely risk averse for small projects, but only modestly risk averse for large projects.
    - Behave as if he is only modestly risk averse for projects that hedge different states of the world.

The last effects show that aversion towards Knightian uncertainty is qualitatively different from high risk aversion, as well as from extra risk premia that arise when full diversification is not possible.

There is an extensive literature on possible reasons for deviations from the standard NPV rule. We mention a sample: For venture capital, high discount rates could be required to compensate venture capitalists for taking an active role in management, to adjust for biases in financial projections from entrepreneurs, or alternatively to reflect that the cash flow estimates are conditional on the project being successful (Sahlman 1990). The active (both financial and managerial) role of venture capitalists can be explained by incentive problems (Admati and Pfleiderer 1994). Difficulties to achieve diversification (leading to a priced idiosyncratic risk), and the illiquidity of venture capital investments will also lead to higher required discount rates (Sahlman 1990).

Within general capital budgeting, there are two main streams of literature that explain high hurdle rates. The first explanation relies on incentive problems that arise when the manager has more information than the owner (Harris, Kriebel, and Raviv 1982, Antle and Eppen 1985, Antle and Fellingham 1995, Harris and Raviv 1996). In Antle and Eppen (1985), the existence of organizational slack induces the owner to set a higher hurdle rate than the cost of capital. In Harris and Raviv (1996), it is the manager's preference for empire building that leads to high hurdle rates.

The second explanation for high hurdle rates relies on real options theory (Berk 1999, Ingersoll and Ross 1992, Pindyck 1991). When a project competes with itself at a later date, real options theory implies that a positive NPV is not enough to ensure that a project should be undertaken. In Antle, Bogetoft, and Stark (1998), a model with both organizational slack *and* options to wait is developed.

Models that are based on learning/parameter uncertainty are close in spirit to our approach. Such models have for example been used to explain abnormal returns for certain assets, see e.g., Klein and Bawa (1976), Klein and Bawa (1977), Barry and Brown (1984), and the recent paper by Lewellen and Shanken (2002). Both approaches based on parameter uncertainty and on Knightian uncertainty begin with the idea that true probability distributions are not known, and results similar to ours would arise with a model based on parameter uncertainty. Our non-Bayesian approach is closer to ideas proposed in literature on strategic decision making under high uncertainty (e.g., Courtney, Kirkland, and Vignerie (1997)), and might be more appropriate in situations where uncertainty is so high that it is difficult to define a Bayesian prior over parameter values.

To our knowledge, this is the first paper to apply Knightian uncertainty to real investment decisions, and to show the implications on requirements of net present value and internal rate of return in investment decisions. Within financial economics, Knightian uncertainty has so far been suggested to explain several puzzles and anomalies. Boyle, Uppal, and Wang (2003) explain the own-equity effect. Epstein and Wang (1994) suggest that the equity premium puzzle and possibly the excess volatility puzzle can be explained within a framework with Knightian uncertainty. Dow and da Costa Werlang (1992) and later Mukerji and Tallon (2001) show how incomplete markets can arise.

From a modeling perspective, Knightian uncertainty introduces an extra level of freedom and complexity in the representation of decision makers' preferences. In this paper we aim to be as general as possible and show effects that arise with a minimum of specification. A future goal would be to test the predicted effects within a well-specified parametric framework. Such a specification is for example developed in Boyle, Uppal, and Wang (2003) in testing the role of Knightian uncertainty for the own-equity effect. Models that explain high hurdle rates with incentive problems on the other hand are more specified over the decision makers' preferences. However, such models have a lot of freedom in the set-up of the game played between the manager and the owner. This freedom can for example lead to predictions of both underinvestment and overinvestment, depending on the situation, as in Harris and Raviv (1996). On the contrary our model will always lead to underinvestment. On the other hand, real options theory is strongly specified and, e.g., always leads to lower investment levels compared with the standard NPV rule. However, we will argue that in many situations, e.g. within venture capital, explaining high hurdle rates with real options theory would be a stretch.

To summarize, we believe that our model offers a plausible explanation for high hurdle rates in investment situations with high uncertainty. The theory of Knightian uncertainty provides tractable, formal models that are based on rigorous decision theoretic foundation, and the results fit well with the high hurdle rates observed, e.g., within venture capital and catastrophe insurance. Furthermore, the concept of Knightian uncertainty fits naturally into the literature on strategic decision making under high uncertainty.

To model decision making under Knightian uncertainty, we will use an intertemporal version of the multiple priors expected utility (MEU) model by Gilboa and Schmeidler (1989) (also known as the Maxmin Expected Utility model). In the MEU model, the decision maker is allowed to have multiple probability distributions and his MEU is the minimum expected utility over these. There are two issues with the MEU model that carry over to our model: First, it gives no indication of *how* the multiple probability distributions should be chosen. In fact, it does not even separate between uncertainty and *aversion* towards uncertainty (compare with *risk*, which is decided by the form of the probability distribution of outcomes and *risk aversion*, which is the pointwise curvature of a decision maker's utility function). Second, it does not take into account any information about the *likelihood* of different probability distributions – they are all given the same weight. For the qualitative results in this paper, these issues are less important. The advantage of the MEU model is that the multiple priors have an intuitive understanding, and that it rests on rigorous decision theoretic foundation. We believe that the effects presented in this paper are inherent and we would expect similar effects to arise with other methods of modeling Knightian uncertainty.

The rest of this paper is organized as follows: In Section 1, we review the multiple priors expected utility model by Gilboa and Schmeidler (1989), and motivate the use of multiple priors with a simple example. In Section 2, we study a two period example and show which effects arise when using a MEU model for investment decisions. In Section 3, we show that the effects hold under general conditions, and make some observations about how to test the model empirically. Finally, in Section 4, we conclude with a discussion of further implications of the model.

1 MEU optimizing decision makers

Our approach to incorporating Knightian uncertainty into decision making follows the Gilboa and Schmeidler max-min theory. With this approach, the decision maker is allowed to use different priors when considering different choices. To give some intuitive justification for this approach, let us consider the following example:

1.1 Intuitive rationale for using multiple priors

Example 2 : Production investment

In a situation of discontinuous technological transition, the owner of a production plant chooses between closing down operations at zero cost or investing \$100 million to rebuild the plant to produce a new product, A. The decision must be made immediately and can not be reversed. The owner knows that there will be a huge demand, either for product A or for a complementary product, B, depending on factors that are outside the owner's control. The competing Betamax and VHS technologies could serve as an example of such a situation.

We stress that there is *no* way for the owner to influence the outcome. The owner's company might, e.g., be a small supplier to one of a few very large competing companies in the end product market. These large companies have the market power and their strategic moves will ultimately decide which technology "wins".

If the owner chooses to rebuild, then if A wins, net revenues will be \$250 million, but if B wins, net revenues will be zero. The decision tree is shown in Figure 1.

[Figure 1 about here.]

The owner has consulted several experts. However, as the products are untried, the information he has received has been diverse, from there being a 75% chance that product A will win, to there being an 80% chance that product B will win. The owner, who would have preferred to know the probabilities for sure, decides to be somewhat conservative and assumes the probability of A winning to be 30%. Therefore, he decides to avoid the investment and close down operations.

Now, suppose the owner instead was choosing between producing product B (with the same investment and potential net revenues), or closing down operations. Would we really expect him to choose the same probability assessment (i.e., a 70% chance for product B to win)? It seems plausible that he would reverse the assessment and once again choose to close down operations.

Thus, we could have a situation with an "A-firm" and a "B-firm", with identical owners, who have identical information, where both owners choose to close down operations, even though both investments can not have nonpositive NPV.

□

The uncertainty in this example is different than increased risk, as can be seen in Figure 2. In the left part of the figure, we see how increased risk could be introduced by changing the spread of *outcomes*. It is represented by adding and subtracting a small $\epsilon > 0$ to the outcomes. In the right part, we see how uncertainty is introduced by changing the *probabilities* for the different states. Note that, to a first order approximation, the risk (variance) of the right lottery is constant when ϵ is added. On the contrary, it is proportional to ϵ for the left lottery. We shall see that these differences have both quantitative and qualitative implications.

[Figure 2 about here.]

There are several reasons why decision makers might prefer lotteries that are purely risky compared with those that are also uncertain. One obvious reason is that more information is given for lotteries where probabilities are known, and we would intuitively expect rational decision makers to prefer as much information as possible. Another reason could be that in uncertain situations, decision makers fear that there might be someone, "playing the other side of the game", who can influence the probabilities in the "wrong" direction. In the Ellsberg example, it could be someone who change the distribution of balls after the decision maker has chosen a lottery. A third reason could be that decision makers want to avoid situations where they in retrospect might feel that they "should have known better". For lotteries where a decision maker is confident about probabilities, bad outcomes depend on bad luck, objectively outside of the decision maker's control. For lotteries where he is not confident, a bad outcome could mean that the decision maker should have evaluated information differently, putting higher weight on some information, etc. It is then not surprising if a decision maker prefers the first type of lotteries.

We should ask ourselves: *If there is unrealized, or even destroyed, value by Knightian uncertainty, why is it not arbitraged away?* In the production investment example, why does not an arbitrageur buy both the "A-firm" and the "B-firm" and avoid uncertainty altogether? For the model to make sense there must be barriers to such arbitrage strategies. There are a multitude of factors that could work as such barriers. Factors that hinder diversification against risk will typically also hinder hedging against Knightian uncertainty. We give two examples:

First, high cost of information could be an important barrier to hedging. For example, for the venture capital industry, cost of *acquiring* information is not only high, but information is also costly (i.e., time consuming) to *digest*. A high degree of trust can remove this barrier, i.e., if investor A trusts investor B, he might not need to analyze B's information to take part in an investment. However, arguably, trust works best in small and tight networks, which are exactly the type structures we see in the small private partnerships in venture capital firms.

Second, regulatory constraints could pose a barrier to hedging. Broadband roll-out in several European and Asian countries might serve as example[7]. The business case for broadband seems to be solid in most European and many Asian countries, throughout the value chain (broadband access, content/services). However, broadband has not been rolled out as rapidly as was predicted, which might be explained by regulatory constraints on industry structure. Several technologies can be used to provide broadband (cable, DSL, satellite, fiber, wireless). These technologies are complementary (more users of one technology will imply fewer in others). Moreover, while there will clearly be high demand for future broadband services, it is *not* clear how the split of revenues will be divided between access and content providers.

However, in many countries regulatory constraints make it impossible for companies to hedge these uncertainties. In Israel for example, the possibility for cross-ownership between cable and DSL providers is highly restricted, as are the possibilities for mergers within a technology. Furthermore, there are regulatory constraints on the possibilities to expand in the value chain: cable providers for example, are not permitted to join up with content providers. These type of regulatory constraints are common in media related industries, as the public service dimension is strongly protected. Thus, the uncertainties in which technology will dominate, as well as in the split of revenues in the value chain, can not be hedged. Under these circumstances, the slow broadband roll-out in Europe and many Asian countries is not surprising.

A couple of comments on the above examples: First, barriers to hedging are costly. In the production investment example, value is destroyed when neither firm invests. Thus, eliminating such barriers can potentially create value. Second, the barrier in the broadband industry points to a potential agency problem. It is the management of the companies that can not hedge, not the owners (the shareholders). We will elaborate on these comments further on.

There are some key properties of the production investment example, which the approach in this paper rests upon:

1. *It is a one-shot decision.* The owner does not have an opportunity to wait with the investment. This could for example be the case in a competitive situation with a first mover advantage.

2. *The investment is irreversible.* Once made, the investment can not be reversed, at least not without significant extra costs.

3. *There is Knightian uncertainty.* The owner does not have enough information to form a confident assessment of the probability distribution. If statistical methods are used, this would typically be the case if historical data is limited.

4. *There is hedging.* A good state of the world for one investment (product A wins) is bad for the other investment and vice versa.

It is enlightening to compare the previous examples with real options theory (Berk 1999, Ingersoll and Ross 1992, Pindyck 1991). Real options theory also provides a modification of the standard NPV rule such that a positive NPV is no longer sufficient to ensure that a decision maker undertakes a project. It has for example successfully been used to explain investment behavior in land development (Quigg 1993).

As in the production investment example, real options theory assumes irreversible investments. However, what drives the results in real options theory is the project competing with itself at a later date, i.e., the value embedded in the option to wait with an investment. This is fundamentally different from our situation, as we are assuming one-shot games.

In situations of one-shot type, using real options theory to explain deviations from the standard NPV rule will be a stretch. For example, in a competitive situation where there is a first mover advantage, the option value of waiting to invest will typically diminish, as shown in Grenadier (2002)[8]. Also, for venture capital investments, it would be difficult to use real option theory: An investor deciding whether to invest seed capital in a startup will typically not have the opportunity to wait and invest in the project at a later date. It could of course be argued that there is an option for an investor – not between investing in a project immediately or at a later date – but between investing immediately or waiting and hoping for an even better project to arrive. However, at the aggregate level this can not explain high required returns. In an equilibrium there should be enough capital to support all projects with positive NPV. This argument should especially hold true during the 1990's, when risk capital was abundant.

1.2 Gilboa and Schmeidler's axiomatic decision model

We give a brief description of the axiomatic foundations to multiple priors expected utility (For a detailed description, see Appendix A). The MEU model follows the Anscombe and Aumann (1963) framework, by assuming that a decision maker chooses between different acts[9], where the outcome is decided in a two-stage process. The first stage is a horse lottery, where the outcome depends on the realized state of the world. Depending on the outcome of the first stage, a roulette lottery is played (which could be a trivial lottery with no risk), and the decision maker then receives the outcome. The structure of the lottery is shown in Figure 3.

[Figure 3 about here.]

The decision maker's problem is to choose an optimal act among a set of acts, L. He is assumed to have a preference relation, $\succeq \subset L \times L$, satisfying the standard axioms of *weak order, continuity, monotonicity* and *nondegeneracy.* In the SEU model, he is also assumed to satisfy the *independence* axiom.

The key difference of the MEU model compared to the SEU model is allowing for "Ellsberg type" hedging against uncertainty (remember how two uncertain lotteries were hedged to one certain in the NRL lottery), by weakening the independence axiom. The weakened independence axiom is the so called *C-independence* (or *certainty-independence*) axiom. In addition, the decision maker is assumed to satisfy an axiom of *uncertainty aversion.* The resulting theorem allows for the decision maker to have a whole set of probability assessments for the horse lottery, which he minimizes expected utility over. The following MEU theorem replaces the classical von-Neumann Morgenstern expected utility theorem:

Theorem 1.1 MEU theorem (Gilboa & Schmeidler)

Assume that a decision maker satisfies the axioms of weak ordering, continuity, monotonicity, nondegeneracy, C-independence and uncertainty aversion. Then, there is a closed, convex *(nonempty) set of probability distributions over the horse dimension, C, which we call the core, and a utility function, u, such that*

$$f \succeq g \Leftrightarrow \min_{\mu \in C} \int \int u(x) df(s) d\mu \geq \min_{\mu \in C} \int \int u(x) dg(s) d\mu.^{10} \tag{5}$$

Here, $df(s)$ is the probability measure induced by the roulette lottery played if the state of the world turns out to be s, and the act chosen is f, and similarly for $dg(s)$.

The utility function is unique up to an affine transformation.

Thus, each μ in Theorem 1.1 is a probability distribution over the outcomes of the horse lottery, and decision makers satisfying the conditions in the theorem are allowed to have whole sets of probability distributions over this dimension. We call such decision makers *MEU optimizers*, and we define multiple priors expected utility, with a specific core,

$$U(f|C) \stackrel{\text{def}}{=} \min_{\mu \in C} \int \int u(x) df(s) d\mu. \tag{6}$$

The Ellsberg paradox is now easily resolved in an MEU framework (as is the product investment example):

Example 3 : Ellsberg three-color example – continued

A decision maker might associate the following core with the different events:

$$C = \{(\mu_{\text{red}}, \mu_{\text{blue}}, \mu_{\text{yellow}}) : \mu_{\text{red}} = 1/3, \mu_{\text{blue}} = 1/3 - \xi, \mu_{\text{yellow}} = 1/3 + \xi,\ \xi \in [-1/6, 1/6]\}, \tag{7}$$

and utility $u(\$10) = 10$, $u(\$0) = 0$. The core is shown in Figure 4. In this case, the decision maker could associate the probability $(1/3, 1/6, 1/2)$ with RL and BL, and $(1/3, 1/2, 1/6)$ with NRL and NBL. Thus, it would be possible to have the following ranking:

$$\text{NRL} \succ \text{NBL} \succ \text{RL} \succ \text{BL}, \tag{8}$$

and the paradox is resolved.

□

[Figure 4 about here.]

2 Investing under Knightian uncertainty – A two period example

Let us consider a two period example, shown in Figure 5, with investments at time $t = 1$ and payoffs at time $t = 2$ that are immediately consumed (Details for this example are given in Appendix B).

[Figure 5 about here.]

The decision maker has a logarithmic utility function:

$$u(x) = \log(x). \tag{9}$$

His initial endowment is unity. There is one *riskfree* project, p_0 and two projects, p_1 and p_2, which are risky *and* uncertain, but which we simply call *risky* projects. The horse dimension is modeled by two states of the world: one "low" and one "high", $S = \{s_{\mathrm{L}}, s_{\mathrm{H}}\}$, and there is also a roulette dimension, with states $Q = \{q_{\mathrm{L}}, q_{\mathrm{H}}\}$. The total state space is $V \stackrel{\mathrm{def}}{=} S \times Q$, and $v \in V$ is a specific state. The return of the riskfree project is normalized to unity. The payoffs in the different states are shown in Table 1.

[Table 1 about here.]

The decision maker's problem is to optimize his multiple priors expected utility of second period consumption by investing his money in the first time period. We assume that the decision maker is allowed to "short-sell" the projects[11].

To begin with, we assume that the two uncertain projects are inseparable, i.e. the decision maker can only invest in the two projects combined. The price for this investment is P. Thus, the decision maker chooses between two projects with returns, r, shown in Table 2.

[Table 2 about here.]

The roulette probabilities are objectively known, $\mathbb{P}(q_{\mathrm{H}}) = \mathbb{P}(q_{\mathrm{L}}) = 1/2$, and are independent of the horse probabilities.

We will now see that when uncertainty is introduced, the decision maker acts in a similar manner as he would if cost of capital increases.

2.1 Decision maker acts as if cost of capital increases when uncertainty increases

We begin with studying the decision maker's choice for the uncertain project of Table 2 as a function of price, for some different degrees of uncertainty. With no uncertainty, the probabilities for different states of the world are assumed to be, $\mathbb{P}(s_{\mathrm{H}}) = 19/20$, $\mathbb{P}(s_{\mathrm{L}}) = 1/20$. However, with increased uncertainty, there will be a core of probabilities:

$$C_\xi = \Big\{(\mathbb{P}(s_{\mathrm{H}}), \mathbb{P}(s_{\mathrm{L}}))\Big\} = \Big\{(19/20 + \alpha, 1/20 - \alpha) : -\xi \times 19/20 \leq \alpha \leq \xi \times 1/20\Big\}. \qquad (10)$$

Here, $\xi \in [0, 1)$ decides the level of uncertainty, an increased ξ implying increased uncertainty. If $\xi = 0$ we have the SEU case.

Now, this is not the only way uncertainty can be increased: There are many ways of creating nested subsets of intervals, and in higher dimensions there will be innumerable ways of increasing uncertainty. However, as will be shown in Section 3, the results hold true regardless of *how* uncertainty increases. The decision maker's demand for the risky project as a function of P is derived by solving the MEU maximization problem:

$$\alpha(P, C_\xi) = \arg\max_{\alpha} \min_{\mu \in C_\xi} \int \int \log(1 + \alpha(r(v)/P - 1))dp\, d\mu. \qquad (11)$$

First, as long as there is positive demand for the risky project, *the demand for the risky project decreases when uncertainty increases.* In Figure 6, we see the demand for the risky project as a function of ξ for prices $P = 0.8$, 0.9, and 1.0. In each case, as ξ increases, the demand decreases to zero and then stays there, never becoming negative. Thus, in times of increased uncertainty we would expect investments to decrease and more capital to be invested in the riskfree project.

[Figure 6 about here.]

For the second effect, we move onto studying the whole set of projects spanned by Table 1, not only the restriction to $p_1 + p_2$. We therefore allow the decision maker to choose any combination of projects, investing α_i in project p_i, as long as it meets the budget constraint

$$\alpha_0 + \alpha_1 + \alpha_2 = 1. \tag{12}$$

We assume that the price for p_1 is 1 and the price for p_2 is 1/100, and study in which regions the decision maker would prefer the investment over allocating all money in the riskfree project. Thus, the decision maker is *not* given the choice to invest in a fraction of the project: he either invests everything or nothing in the risky project. This type of situation is closer to real world situations where investments are typically indivisible. We call the set of projects that the decision maker would prefer at least as much as investing in the riskfree project, *the set of preferred projects* (or simply, *the preferred set*), and we denote it by P_0. The sets of preferred projects for different levels of uncertainty are shown in Figure 7. For later references, we have also included the set of perfectly hedgeable states (the dotted line, Γ), i.e., the states for which the decision maker is indifferent if uncertainty increases or not. This happens when expected utility from s_{H} is equal to expected utility from s_{L}. Within the set of perfect hedges, MEU coincides with SEU. However, this set will typically be a small subset of the total sets of projects. We see that *the set of preferred projects strictly decreases when uncertainty increases.*[12]

Finally, we study how increased uncertainty affects required internal rates of return. Of course, we should not include investments in the riskfree project, as it always has (excess) $IRR = 0$. We therefore exclude the riskfree project and look at the investments satisfying

$$\alpha_1 + \alpha_2 = 1. \tag{13}$$

We study the lowest IRR of a project in the set of preferred projects, and plot it as a function of uncertainty[13]. The results are shown in Figure 8. We see that the *required IRR is a strictly increasing function of uncertainty.* Thus, in a situation where uncertainty increases, the decision maker will require a higher IRR to consider taking on a proposed project.

[Figure 7 about here.]

[Figure 8 about here.]

The effects are interesting from the venture capital perspective. If the model is correct, we should not be surprised to see high required rates of return in uncertain industries.

2.2 Increased uncertainty decreases power of NPV rule for decision maker

The results of Section 2.1 depend on the decision maker being both risk averse and uncertainty averse. It is fair to ask how we can be sure that it is uncertainty aversion that is driving the results. *Would we get the same results if the decision maker were risk neutral?* The answer is, in principle, *yes*, but the notation will be more cumbersome. With risk neutral decision makers, preferred sets are no longer bounded and demand curves become infinite. Instead, the preferred sets are cones: $p \in P_0, \alpha \in \mathbb{R}^+ \Rightarrow \alpha p \in P_0$. The demand will either be $+\infty$, 0, or $-\infty$. However, the results are the same: With increased uncertainty, the set of preferred projects strictly decreases, regions with no demand increase, and required IRR strictly increases. Thus, the effects are not driven by risk aversion.

To study how well the NPV rule works for the decision maker under increased uncertainty, we assume that the decision maker's risk aversion agrees with what is implied by the NPV rule, i.e., we assume that the payoff structure of Table 1 takes the discount factor into account. Under the discounted payoff structure, we should calculate as if the decision maker is risk neutral. The NPV rule then implies that a project should be undertaken as long as $\alpha_1 > 0$.

However, under Knightian uncertainty, this will be a necessary but no longer sufficient condition for the decision maker to undertake a project. The fraction, F, of all NPV positive projects that also have positive MEU as a function of uncertainty is shown in Figure 9.[14] We see that *the fraction is a strictly decreasing function of uncertainty.* Moreover, it decreases quickly for low uncertainties. Thus, in uncertain situations, the decision maker must increasingly use supplemental rules for deciding whether to undertake a project or not.

The payback method could for example have a role to play as a supplement to the NPV rule in uncertain environments. If we believe that uncertainty increases with the time horizon of payoffs, projects with short payback times will be more robust against uncertainty than projects with long payback times. This effect is above and beyond the implicit higher risk of longer time horizons built into the adjusted discount factor. It is of course not possible to see this effect in the two period example.

[Figure 9 about here.]

2.3 Decision maker behaves differently under uncertainty than under pure risk

The effects shown so far are similar to effects implied by risk aversion, which could be used as an argument against models with Knightian uncertainty. *Why complicate things if nothing new is gained?* The simple answer to this question is that risk aversion is already incorporated into the NPV rule through the discount rate. We are explaining how *excessive* risk aversion can be incorporated through nonstandard behavior over *probabilities*. We mention three reasons why such behavior can arise:

1. Knightian uncertainty is present and decision makers are uncertainty averse, as assumed in this paper.

2. Decision makers are more risk averse than permitted by the standard NPV rule. However, they have been trained to use NPV/discounted cash flow analyses (e.g., in MBA programs). Furthermore, there are "objective" ways of determining the right discount rate, e.g., by using the CAPM. The only free parameters in the model are the probabilities, and to reach an answer that is closer to their personal preferences, they use conservative estimates.

3. Decision makers are not rational, and have nonlinear assessments over the probability space.

Of these three explanations, I find the first most appealing, as it is based on rational behavior. However, the key point of the MEU model is that decision makers use conservative measures for probabilities, whatever the reason might be. Moreover, MEU behavior fundamentally differs from SEU behavior. It is not just an increase in risk aversion. To show this, we begin by studying the demand curve for the risky project, when the decision maker decides how much to invest in the project of Table 2. In Figure 10, we see the demand curve for three different levels of uncertainty, ξ. The left solution is with no uncertainty. This case reduces to the SEU case. The key difference for the two cases with uncertainty is that the demand curves have "kinks". These arise from the investor switching probabilities in his optimization, when changing from considering going long to going short in the risky project. The kinks are similar to what is found in behavioral models, attributed to loss aversion, see Kahneman and Tversky (1979).

[Figure 10 about here.]

The kinked demand curves give an indication of how misleading the risk aversion measure will be in the MEU model. Relative risk-aversion:

$$RRA \stackrel{\text{def}}{=} -\frac{xu''}{u'}, \tag{14}$$

can still be defined over the roulette dimension of the model, but if we fail to separate the roulette and the horse dimensions, we will get strange results. For example, in the SEU case, we can estimate the decision maker's RRA at a certain point without knowing his subjective probability assessments, by finding his certainty equivalent to low-risk lotteries. We ask the decision maker which π_0 makes him indifferent between the certain payoff of $1-\pi_0$ and a horse lottery with positive or negative payoff of ϵ:

$$s_{\text{H}} \Rightarrow 1-\epsilon, \qquad s_{\text{L}} \Rightarrow 1+\epsilon, \tag{15}$$

and also between $1-\pi_1$ and the horse lottery

$$s_{\text{H}} \Rightarrow 1+\epsilon, \qquad s_{\text{L}} \Rightarrow 1-\epsilon. \tag{16}$$

In the SEU setup, we have the identities

$$\mathbb{P}(s_{\text{H}}) + \mathbb{P}(s_{\text{L}}) = 1, \tag{17}$$

$$u(1-\pi_0) = \mathbb{P}(s_{\text{H}})u(1-\epsilon) + (1-\mathbb{P}(s_{\text{L}}))u(1+\epsilon), \tag{18}$$

$$u(1-\pi_1) = \mathbb{P}(s_{\text{H}})u(1+\epsilon) + (1-\mathbb{P}(s_{\text{L}}))u(1-\epsilon). \tag{19}$$

By Taylor expanding u around unity, we can use (17-19) to derive that for small ϵ:

$$\text{RRA} = \frac{\epsilon(\pi_0+\pi_1)}{2\epsilon^2 - \pi_0^2 - \pi_1^2} + \mathcal{O}(\epsilon). \tag{20}$$

However, in the MEU setup, the decision maker will use different probabilities in (18) and (19), leading to

$$\text{RRA} + 2\xi\mathcal{O}(1/\epsilon) = \frac{\epsilon(\pi_0+\pi_1)}{2\epsilon^2 - \pi_0^2 - \pi_1^2} + \mathcal{O}(\epsilon). \tag{21}$$

Thus, as ϵ approaches zero, the observed "risk aversion" will be unboundedly overestimated.[15] The effect above is straight to the core of the MEU model: With Knightian uncertainty present, MEU models separate two aversions (one over pure risk, which is closely related to the diminishing marginal utility of wealth and one over uncertainty, which reflects the preference for hedging in situations when one does not know the probabilities for different states of the world), which SEU models treat as one.

Let us refine the study of this effect, to see how MEU and SEU optimizers differ. By studying the set of preferred projects under increased risk aversion versus increased uncertainty aversion, we can see how MEU optimization leads to different preferred sets than SEU optimization. We use the set of projects spanned by Table 1. The logarithmic utility function has $RRA \equiv 1$. Now, assume that the decision maker has $RRA = 1 - \gamma$ instead, with $\gamma < 0$, corresponding to the utility function

$$u(x) = \frac{x^\gamma - 1}{\gamma}. \tag{22}$$

In Figure 11, the set of preferred projects is shown for $\gamma = -0.2$, and compared to the example with uncertainty, with $\xi = 0.2$. We see that there is a region (A), in which the uncertainty averse decision maker chooses to invest, whereas the (more) risk averse decision maker chooses not to invest. Also, there are regions (B) where the situation is the opposite: The risk averse decision maker invests, but not the uncertainty averse decision maker. The B regions lie closer to the riskfree project (the origin), and the closer we get, the higher the implied risk aversion has to be, for an SEU optimizing decision maker to behave like an MEU optimizing decision maker. For example, for an SEU optimizing decision maker not to invest in $(\alpha_1 = 0.57, \alpha_2 = 0.03)$, he has to have an RRA of 4 or higher. Close to the perfect hedge however, at $(\alpha_1 = 3.1, \alpha_2 = 0.16)$, even a small increase in risk aversion will make the SEU optimizer avoid projects that the MEU optimizer will invest in.

[Figure 11 about here.]

Thus, *for large investments with a high degree of hedging, the risk aversion of an MEU optimizer will seem modest when treated as an SEU optimizer*, whereas *for small investments with a low degree of hedging, the risk aversion will seem high.* The first type of investments are close to what might be thought of as "no regret moves", whereas the second type are "small" projects with limited opportunities (and costs). This effect is in line with Ross (1986), who noted that higher hurdle rates are used for small projects than for large within capital budgeting in a study of budgeting practices for twelve manufacturing firms.

The previous argument also holds for other parameter changes in the SEU model than increased risk aversion. For example, if we change the mean or variances of the roulette lotteries, this leads to similar changes in the set of preferred projects as in the SEU case (smaller change for small projects and larger change for hedgeable projects than under MEU optimization). Thus, MEU behavior is really fundamentally different than what is seen in the von-Neumann Morgenstern framework.

We now generalize the results of this example to multiple time periods, general utility functions and general projects.

3 Investing under Knightian Uncertainty – Theoretical Results

3.1 A multiperiod investment model

We generalize the Gilboa & Schmeidler model to an intertemporal setting (For details, see Appendix C). The decision maker's problem is to choose an optimal project to invest in, among a set of projects. Each project gives the outcome of a roulette lottery in each time period, where the roulette lottery depends on the realized state of the world.

We assume that the decision maker can not freely reallocate investments over time, but has to stick to the project once the investment decision is made (irreversibility).

3.1.1 The decision maker's investment decision

There are T time periods, and a state space in each time period, S_i, $i = 1, \ldots, T$. For simplicity, we assume that every possible combination of the state of the world is possible (i.e., has a positive probability of occurring), so the total state space is

$$\mathcal{S} \stackrel{\text{def}}{=} S_1 \times S_2 \times \cdots \times S_T. \tag{23}$$

Each S_i is assumed to be finite, and the total number of possible states is N. We always assume that there are at least two states of the world, $N \geq 2$.

The decision maker's problem at time $t = 0$, is to choose a maximal element from a set of *projects* that generate cash flows from time period 1 to T. A project is a function that, in each state of the world, in each time period realizes the outcome from a specific roulette lottery. We call the intertemporal set of roulette lotteries, $\mathcal{Y} \stackrel{\text{def}}{=} Y^T$, where Y is the space of one period lotteries. An intertemporal project is then a function:

$$p : \mathcal{S} \to \mathcal{Y}. \tag{24}$$

There is a riskfree project, p_{riskfree}, which generates zero (excess) return in each time period, regardless of the realized state of the world.

Now, let us assume that there are n projects that the decision maker chooses between:

$$\mathcal{P} \subset \mathcal{Y}^{\mathcal{S}} = \{p_1, p_2, \ldots, p_n\}. \tag{25}$$

We also assume that the riskfree project is not included in $\mathcal{P}$, and that n is finite. Unless otherwise stated, we assume that there are at least two risky projects, $n \geq 2$. The optimization problem over $\mathcal{P}$ is the simplest, and the set of optimal projects will of course always be non-empty and finite, as long as the decision maker has a complete, transitive ordering over projects. When we want to include the riskfree project in the decision problem, we use the following notation: $\bar{\mathcal{P}} \stackrel{\text{def}}{=} \mathcal{P} \cup \{p_{\text{riskfree}}\}$. We define P^* to be the set of optimal projects, and the *set of* *preferred projects*, P_0 to be the (possibly empty) set of projects preferred as least as much as the riskfree project.

There are also richer sets of projects of interest: The first is if we assume that the decision maker can divide his money between the projects in $\mathcal{P}$. Thus, he invests a fraction, α_i in project p_i under the budget constraints:

$$\sum_{i=1}^{n} \alpha_i = 1, \qquad \alpha_i \geq 0, \ i = 1, \ldots, n. \tag{26}$$

We call this set of projects $L_\Delta(\mathcal{P})$ (and $L_\Delta(\bar{\mathcal{P}})$ respectively, depending on whether the decision maker is allowed to invest money in the riskfree project, in which the sum goes from 0 to n in (26)). Finally, we will be interested in the even larger sets of projects, $L_/(\mathcal{P})$, and $L_/(\bar{\mathcal{P}})$, where the decision maker is allowed to short-sell projects, i.e., to invest a negative amount of money in projects. The budget constraint then is

$$\sum_{i=1}^{n} \alpha_i = 1. \tag{27}$$

Although this is somewhat removed from what we would think of as real situations, it simplifies the analysis as we do not need to worry about boundaries. In a real world situation, under the natural additional assumption that the set of preferred projects lie in the non-shorting region, the theorems will hold true even if we replace $L_/$ with L_Δ. We assume that when investing in multiple projects, the roulette dimension of the outcome is independent for each project.

The MEU model generalizes to the multiperiod case and ensures the existence of a probability core, $\mathcal{C}$, a utility function, $\mathcal{U}$, and a maxmin theorem similar to Theorem 1.1, as shown in Appendix C. For a specific project, p, we write its MEU under core $\mathcal{C}$ as $U(p|\mathcal{C})$.

3.1.2 Assumptions over roulette dimension

We make the standard assumption that the utility function is time separable:

Assumption 3.1 Time-separable utility over roulette dimension

$$\exists u : \mathbb{R} \to \mathbb{R} : \textit{ such that } \mathcal{U}(x_1 \times x_2 \times \cdots \times x_T) = \sum_{j=1}^{T} \rho^{-j} u(x_j), \textit{ for some } \rho > 0.^{16} \tag{28}$$

where x_j is the outcome in time period j.

We also need:

Assumption 3.2 Standard assumption on u

1. $u(0) = 0$,
2. *u is strictly concave, strictly increasing, and twice continuously differentiable,*
3. $\lim_{x \to +\infty} u(x)/x = 0$.

3.1.3 Increasing uncertainty

We wish to define what it means for a situation to be more uncertain than another. In the one dimensional case it was natural to view increased uncertainty as the interval covered by the core becoming larger. We carry over this notion to the general case, using a definition that stresses that uncertainty is a function of the degree of information a decision maker has. We thus introduce an abstract information set, $\mathcal{I}$, which symbolizes what the decision maker "knows" at time $t = 0$, and $\mathcal{C}(\mathcal{I})$, which is the decision maker's core under information set $\mathcal{I}$. The definition then takes the form:

Definition 3.1 *Consider two information sets, $\mathcal{I}_1$ and $\mathcal{I}_2$: Given a set of MEU-optimizing decision makers, $\mathcal{I}_2$ is said to be weakly more uncertain than $\mathcal{I}_1$, if for any decision maker, $\mathcal{C}(\mathcal{I}_1) \subseteq \mathcal{C}(\mathcal{I}_2)$.*

Thus, any decision maker will have a (weakly) larger core when given information $\mathcal{I}_2$ compared with $\mathcal{I}_1$. The definition of *strictly* more uncertain information set is somewhat technical:

Definition 3.2 *Consider two information sets, $\mathcal{I}_1$ and $\mathcal{I}_2$: Given a set of MEU-optimizing decision makers, $\mathcal{I}_2$ is said to be strictly more uncertain than $\mathcal{I}_1$, if it is weakly more uncertain, and for each decision maker $d(\mathcal{C}(\mathcal{I}_1), \partial\mathcal{C}(\mathcal{I}_2)) > 0$.*

The idea is that the core becomes strictly larger in each direction. The distance function, d, used is the Euclidean distance. The difference between weakly and strictly increased uncertainty, is shown in Figure 12, for the Ellsberg example discussed in the Introduction and Section 1.

[Figure 12 about here.]

These definitions are *operational* in that they are not derived from the decision makers underlying preferences. By working directly on the cores, we avoid losing ourselves in deep questions of eventwise consistency (analyzed, e.g., in Machina and Schmeidler (1992), and Epstein (1999)). The decision theoretic ground for our definition of weakly increased uncertainty is given in Ghirardato and Marinacci (2002). Moreover, our definition of strictly increased uncertainty is along similar lines as proposed in Ghirardato, Maccheroni, and Marinacci (2002).

We also assume that there is an objective measure of uncertainty neutral probabilities. Thus, given an information set, even though each decision maker might have a unique core, they all agree on an objective uncertainty neutral probability. Compare this with risk neutrality where, regardless of individual attitude towards risk, everyone agrees on what it means to be risk neutral.

3.1.4 No arbitrage and set of perfect hedges

Assumption 3.3 No arbitrage

There is no project in $L_/(\mathcal{P})$ that guarantees nonnegative outcomes in all states of the world, and strictly positive outcome in at least one state of the world.

In many cases, there will be elements in $L_{/}(\mathcal{P})$ for which all horse states give the same expected utility. In the example of Section 2, this set was represented in Figure 7 by the dotted line, Γ. We define

Definition 3.3 Set of perfect hedges

The set of perfect hedges, Γ *for a decision maker, is defined as the elements in* $L_{/}(\mathcal{P})$ *(and* $L_{/}(\bar{\mathcal{P}})$*) for which:*

$$\exists c, \forall s \in \mathcal{S} : \int u(x) df(s) = c. \tag{29}$$

Typically, this set will have much lower dimension than n. In the full rank case, the set will be a manifold of dimension $n - N$ if $N \leq n$, and an empty set if $N > n$.[17] However, under the general assumptions we have made on u and the lotteries, there is very little we can say of the rank of the system of equations arising from (29), and in degenerate cases, Γ can cover a substantial part of $L_{/}(\mathcal{P})$[18].

For elements in Γ, increased uncertainty will never change the decision maker's choice, and MEU theory "adds" nothing to SEU theory. It is not the decision maker's attitude towards *uncertainty* that decides Γ. Instead it is a combination of the decision maker's attitude towards *risk*, and the *roulette lotteries.* Moreover, except for p_{riskfree}, elements of Γ are not free from *risk* (the roulette dimension). It is the *uncertainty* (the horse dimension) that is irrelevant in the set.

With these definitions and assumptions, we are now ready to proceed to the main propositions.

3.2 Main propositions

We shall generalize the properties of the example in Section 2, i.e., show how the decision maker acts as if cost of capital has increased, how he will rely less on the NPV rule as uncertainty increases, and how he will choose different types of projects than if he were simply more risk averse. Proofs of propositions are given in Appendix C.

3.2.1 Investment levels decrease when uncertainty increases

For the first effect of Section 2.1, we need a so called normality condition on the expected utility. This is a technical condition, which is formally defined in the appendix, and which says that the derivatives of the expected utility of the roulette lotteries will have the same ordering regardless of how much is invested in the risky project. It will be satisfied, e.g., in cases where negative outcomes do not depend on the state of the world, but positive outcomes do, (as is the case in Section 2). A more detailed discussion of why it is needed is given in the proof.

With these assumptions, the following proposition holds true:

Proposition 3.1 *Consider a decision maker with utility function that satisfies Assumption 3.2, choosing the fraction α to invest in a risky project that permits no arbitrage, and that satisfies the normality condition. Given two information sets, $\mathcal{I}_1$ and $\mathcal{I}_2$, where $\mathcal{I}_2$ is strictly more uncertain than $\mathcal{I}_1$: Call the amount the decision maker invests under $\mathcal{I}_1$ and $\mathcal{I}_2$, α_1 and α_2 respectively. Then,*

(i) If $\alpha_1 = 0$, then $\alpha_2 = 0$.

(ii) If $|\alpha_1| > 0$, then $|\alpha_2| < |\alpha_1|$.

Thus, similar to the example of Section 2, increased uncertainty always moves investments towards the riskfree project.

3.2.2 Fewer projects are undertaken when uncertainty increases

To generalize the second effect of Section 2.1, we need to consider the set of perfect hedges, Γ. In the case that Γ is "small", we have the following result:

Proposition 3.2 *Consider a decision maker with a utility function that satisfies Assumption 3.2, facing a set of projects, $L_/(\bar{\mathcal{P}})$, which permits no arbitrage. Given two information sets, $\mathcal{I}_1$*

and $\mathcal{I}_2$, where, $\mathcal{I}_2$ is strictly more uncertain than $\mathcal{I}_1$: Call the set of preferred projects under $\mathcal{I}_1$ and $\mathcal{I}_2$, P_0^1 and P_0^2 respectively. If P_0^1 has non-empty interior, then

(i) For every $p \in P_0^1$, not belonging to the set of perfect hedges, Γ:

$$U(p|\mathcal{C}(\mathcal{I}_2)) < U(p|\mathcal{C}(\mathcal{I}_1)). \tag{30}$$

(ii) If $\Gamma = \emptyset$, then $d(\partial P_0^1, P_0^2) > 0$.

Here, the distance function, d is the Euclidean distance between the investments (the αs).[19]

As earlier noted, in the general case we know little about Γ. If Γ covers the whole set of preferred projects, then the level of uncertainty is irrelevant. By making additional assumptions on the projects we can ensure that Γ, is not "too large". Intuitively, we would expect Γ to be large if the projects are very similar. To ensure that this is not the case, we define *moment conditions* in Appendix C that, loosely speaking, ensure that there are two projects that have different expected returns in some states of the world.

Furthermore, we know from the previous section that demand for a project satisfying the *normality condition* decreases when uncertainty increases. If there are such projects in the set of preferred projects, then they should become less preferred when uncertainty increases. It turns out that each of these conditions is sufficient to ensure that the set of preferred projects decreases when uncertainty increases. We have:

Proposition 3.3 *Consider a decision maker with utility function that satisfies Assumption 3.2, facing a set of projects, $L_/(\bar{\mathcal{P}})$, which permits no arbitrage, and for which at least one of the following conditions is satisfied:*

(i) $\{p_1, \ldots, p_n\}$ satisfy the moment conditions.

(ii) There is a project in $L_/(\bar{\mathcal{P}})$, that satisfies the normality condition.

Given two information sets, $\mathcal{I}_1$ and $\mathcal{I}_2$, where, $\mathcal{I}_2$ is strictly more uncertain than $\mathcal{I}_1$: Call the set of preferred projects under $\mathcal{I}_1$ and $\mathcal{I}_2$, P_0^1 and P_0^2 respectively. If P_0^1 has non-empty interior, then P_0^2 is a strict subset of P_0^1.

Remark: The moment conditions and normality conditions are quite different. The moment conditions ensure that there are two projects that are (sufficiently) different in their payoff structure. The normality condition ensures that a single project has (sufficiently) different payoffs across the states of the world. To create an example where both fail to be satisfied, and hence the set of preferred projects is not guaranteed to decrease as uncertainty increases, the projects need to be close to degenerate. One case that fails to satisfy both the moment and normality conditions is when all projects give rise to the same roulette lottery in every state of the world.

3.2.3 IRR requirements increase when uncertainty increases

We now generalize the third effect of Section 2.1. Naturally, there is no direct relationship between the quantitative IRR measure and the set of preferred projects. However, there *is* a direct relation between the *lowest* IRR a strictly uncertainty averse decision maker will be willing to accept, and the level of uncertainty. Thus, a high lower bound on the IRR could be a necessary condition for a decision maker to even consider a project.

We need additional assumptions on the projects involved:

Assumption 3.4 Assumptions on projects, $p_1, \ldots, p_n$

1. *Each project has nonnegative expected payoff in each time period, $t = 1, \ldots, T$.*

2. *Each project has a positive IRR.*

3. *There are two projects that have different IRRs.*

The first assumption is needed to avoid situations with multiple IRRs. The second assumption is natural, as we do not expect projects with lower IRRs than the riskfree rate to be considered (although in situations with strong hedging, it could appear). The third assumption is needed to avoid degenerate cases. To avoid the problem of multiple IRRs, we also need to restrict ourselves to elements in L_Δ.

The theorem now takes the form:

Proposition 3.4 *Consider a decision maker with a utility function that satisfies Assumption 3.2, facing a set of projects, $L_/(\mathcal{P})$, that admits no arbitrage and where $p_1, \ldots, p_n$ satisfy the moment conditions and the conditions of Assumption 3.4. Given two information sets, $\mathcal{I}_1$ and $\mathcal{I}_2$, where $\mathcal{I}_2$ is strictly more uncertain than $\mathcal{I}_1$. Call the set of preferred projects under information sets $\mathcal{I}_1$ and $\mathcal{I}_2$, P_0^1, and P_0^2 respectively. Furthermore, assume that the set of perfect hedges, Γ, is empty. If P_0^1 has non-empty interior and lies in $L_\Delta(\mathcal{P})$, then the lowest required IRR of P_0^2 is strictly higher than that of P_0^1:*

$$\inf_{p \in P_0^2} IRR(p) > \inf_{p \in P_0^1} IRR(p). \tag{31}$$

3.2.4 Increased uncertainty decreases power of NPV rule

We saw in Section 2.2 that increased uncertainty decreased the fraction of NPV positive projects that the decision maker invests in. We show this effect in the general case. We need to deal with unbounded sets. We therefore introduce a way of bounding the sets by defining

$$F_{\mathcal{Q}}(A, B) \stackrel{\text{def}}{=} \frac{\lambda(A \cap \mathcal{Q})}{\lambda(B \cap \mathcal{Q})}. \tag{32}$$

Here, $\mathcal{Q}$ is a bounded neighborhood of the origin (under Euclidean distance), and λ is the Lebesgue measure. The fraction of NPV positive projects that the decision maker invests in is defined as

$$F(\mathcal{I}) \stackrel{\text{def}}{=} \lim_{\gamma \to \infty} F_{\gamma B}(P_0(\mathcal{I}), \mathcal{N}), \tag{33}$$

where B is any bounded neighborhood of the origin, $P_0(\mathcal{I})$ is the set of preferred projects under information set $\mathcal{I}$, and $\mathcal{N}$ is the set of NPV positive projects under the uncertainty neutral probabilities. We implicitly assume that the uncertainty neutral probabilities (which we get through the objective measure) do not change when uncertainty increases.

We also assume that the decision maker agrees with the NPV rule in the case without uncertainty. He is thus "risk neutral" in the sense that $u(x) \equiv x$ and all excessive risk aversion is reflected in the discount factor. We now have:

Proposition 3.5 *Consider a decision maker who agrees with the NPV rule in the case with no uncertainty, facing a set of projects, $L_/(\bar{\mathcal{P}})$, where $p_1, \ldots, p_n$ satisfy the moment conditions and the second condition of Assumption 3.4 (All projects have positive NPV). Given two information sets, $\mathcal{I}_1$ and $\mathcal{I}_2$, where $\mathcal{I}_2$ is strictly more uncertain than $\mathcal{I}_1$. Then*

(i) If $F(\mathcal{I}_1) > 0$, then $F(\mathcal{I}_2) < F(\mathcal{I}_1)$,

(ii) If $F(\mathcal{I}_1) = 0$, then $F(\mathcal{I}_2) = 0$.

3.2.5 Decision maker behaves differently under uncertainty than under risk

The following theorem is a generalization of the observation in Section 2.3, that MEU and SEU optimizing decision makers will behave differently for small and hedgeable projects:

Proposition 3.6 *Consider two decision makers, $\mathcal{DM}_1$ and $\mathcal{DM}_2$, with the same information sets, $\mathcal{I}$, and with utility functions that satisfy Assumption 3.2, facing a set of projects, $L_/(\bar{\mathcal{P}})$, that admits no arbitrage and where $p_1, \ldots, p_n$ satisfy the moment conditions and the conditions* *of Assumption 3.4. Assume that $\mathcal{DM}_2$ is strictly* more *uncertainty averse*[20] *and strictly* less *risk averse*[21] *than $\mathcal{DM}_1$. Call their sets of preferred projects P_0^1, and P_0^2 respectively. Moreover, call $\mathcal{DM}_2$'s set of perfect hedges, Γ. Assume that P_0^2 has nonempty interior. Then,*

(i) Except for the riskfree project, any point belonging to $\Gamma \cap \partial P_0^2$, has a neighborhood in which no points belongs to P_0^1.

(ii) There is an open ball around the origin, B, such that

$$F_B(P_0^2, P_0^1) < 1. \tag{34}$$

3.3 Empirical predictions

The MEU model in this paper is consistent with several ways of using the NPV rule in unorthodox ways (and even not using it at all) that have been observed in the real world. It also leads to nontrivial predictions that could, in principle, be tested empirically. It is not within the scope of this paper to perform such tests, but we give some indications to how it may be done.

We mention two types of predictions that arise from the model. The first type is along the lines that decision makers in uncertain situations will *herd* in the sense that the type of projects they choose will be more similar than in situations with pure risk. In the example of Section 2.1, we saw that as uncertainty increased, the projects had to be close to perfect hedges for the decision maker to invest in the project. In Figure 13, we show this effect in a generic picture of the sets of preferred projects under different levels of uncertainty. The "D" region is the set of preferred projects with uncertainty, and the "C" region is the set with no uncertainty. We note that there is one "size" and one "diversity" direction in the figure, represented by the two arrows in the upper right corner. In the size direction, the size of the project (investments, payoffs) increases, but its relative payoff characteristic stays the same. In the diversity direction (which will be high-dimensional in the general case), the required investment is the same, but the payoff characteristic changes. Under increased uncertainty, the spread in size and diversity of the set of preferred projects decreases. This is easily seen for the diversity dimension (the "D" region is much "thinner" than the "C" region). By studying the cone properties of the preferred sets for small and large projects, it can also be shown to hold for the size direction.

If we make the additional assumption that what, in this paper, we have called projects are actually characteristics on the firm level (or equivalently, that each firm is an element of $L_/(\bar{\mathcal{P}})$), then these ideas can be used to predict firm characteristics under different levels of uncertainty. The hypotheses would be along the lines: *All else equal, firms' payoffs will be more similar in situations with higher uncertainty*, and *All else equal, firm sizes will be more similar in situations with high uncertainty.*

The second type of predictions would draw on the agency problem noted in the discussion on stalled broadband markets in Section 1. In uncertain situations, the agency problem introduced by having managers who *can not* hedge uncertainty, and owners who *can*, will be more severe. Therefore, we will expect the manager and owner to be the same person to a higher degree in uncertain industries. The hypothesis would be along the lines: *All else equal, the CEO will own more stocks in the company (s)he works for in situations with higher uncertainty.*

In detailing a test we would need a way to measure the level of uncertainty a firm is exposed to. We would also need to separate between risk and uncertainty. A natural assumption could be that firms within the same industry are exposed to similar uncertainty.

However, how would we assess the *level* of uncertainty within an industry? We might argue as follows: A major source of uncertainty, is lack of historical data. Under stationarity (and some additional) assumptions, uncertainty should asymptotically vanish over time, similar to how a confidence interval decreases when the number of observations increases. In such a situation, the age/maturity of an industry might therefore be used as an inverse proxy for uncertainty. This would permit testing an industry against itself over time. This approach would avoid the issue of separating between risk and uncertainty (if we assume that the risk is the same within an industry over time).

Another approach would be to look at industries in times of structural changes. As previously discussed, drivers of uncertainty within an industry could, e.g., be rapid technological changes or changed regulations. Methodologies similar to those used in event studies could be applied in such situations.

[Figure 13 about here.]

4 Further implications

When Knightian uncertainty is present, the value of being able to hedge increases drastically. It no longer depends solely on risk aversion, but now also on aversion towards uncertainty about probabilities. As we saw in Sections 2 and 3, this can introduce effects of orders of magnitude larger than what is implied by risk aversion.

The value of removing barriers to hedging is higher under Knightian uncertainty. The two examples of such barriers we mentioned in Section 1 were information costs within venture capital and regulations within broadband industries. The barrier for venture capital is inherent, and seems hard to overcome. Other barriers might not be. Regulatory barriers for broadband in several European countries are constructed to protect the public service dimension of media. The results in this paper emphasize the potential costs of such constructed barriers.

Agency problems arise under Knightian uncertainty. Even if the shareholders of a company can hedge, managers who can *not* might choose conservative probability assessments in their investment analyses. This paper further underlines the importance of developing incentive schemes that resolve such issues. Moreover, in situations with Knightian uncertainty, the importance of relating compensation to realized states of the world (e.g., indexed options) is high-lighted.

Investment analyses could be modified to incorporate Knightian uncertainty. Today few, if any, companies use models that directly take Knightian uncertainty into consideration. Rather, it is introduced indirectly through robustness and scenario analyses together with conservative decision rules. However, if management of a company would adopt the MEU model (or some other method to incorporate Knightian uncertainty, e.g., robust control methods), it could formally incorporate the framework into its investment analyses. In this context, robustness and scenario analyses could be viewed as proxies of such an analysis tool.

The model in this paper is consistent with several real world deviations from the NPV rule. Empirical analysis of the predictions lined out in Section 3.3 would further test the validity of the model.

A SEU and MEU approaches to decision making - Background to Section 1

Models with Knightian uncertainty distinguish between two different types of lotteries: those where probabilities are "objectively" known, and those where they are not. Lotteries where probabilities are known are called *roulette lotteries*, whereas those with "uncertain" probabilities are called *horse lotteries*. The distinction was originally made by Knight (1921), and the terminology was introduced by Anscombe and Aumann (1963).

The Gilboa & Schmeidler (GS) approach to decision making follows the Anscombe & Aumann (AA) model in that it works on two-stage lotteries, where the first stage is a "state of the world horse lottery", and the second is an "objective roulette lottery". The GS approach leads to multiple expected utility (MEU) optimization. A closely related model, is the Choquet expected utility (CEU) model introduced in Schmeidler (1989). In the CEU model, the multiple priors are replaced with non-additive probability distributions and Choquet integrals. We choose to use the MEU model, as it is easier to work with sets of priors than with nonadditive probability measures, and because the multiple priors have an intuitively clear meaning.

Following the notation of Gilboa and Schmeidler (1989), we assume a space of outcomes, X. In this paper we will assume that X is a compact set in $\mathbb{R}$ representing possible dollar payoffs (and in the multi-period context, a compact set in $\mathbb{R}^n$, representing payoffs in each time period). The set of roulette lotteries, $Y \subset [0,1]^X$, is defined as

$$Y \stackrel{\text{def}}{=} \left\{ y : X \to [0,1] \,\middle|\, y(x) \neq 0 \text{ for finitely many } x \in X, \text{ and } \sum_{x \in X} y(x) = 1 \right\}. \tag{35}$$

The horse dimension is induced by the set of states of the world, S_0. In this paper, we assume that S_0 is of finite dimension. Σ is an algebra of subsets on S_0, and we can without loss of generality work on the space $S \stackrel{\text{def}}{=} S_0/\Sigma$ of equivalence classes induced by Σ (i.e., $s_0 \cong s_1 \Leftrightarrow \forall F \in \Sigma, s_0 \in F \Rightarrow s_1 \in F$). The generic set of acts is $Y^S \stackrel{\text{def}}{=} \{f : S \to Y\}$. The decisions under consideration will be on a subset of the generic set, $L \subset Y^S$.

We define convex linear operations on elements of Y in the classical "compound lottery" manner: for

$$y_1 : X_1 \to [0,1], \qquad y_2 : X_2 \to [0,1], \tag{36}$$

we have $y \stackrel{\text{def}}{=} \alpha y_1 + (1-\alpha) y_2 \in X_1 \cup X_2$ defined as

$$y(x) = \alpha \hat{y}_1(x) + (1-\alpha)\hat{y}_2(x), \tag{37}$$

where for $i = 1, 2$, $\hat{y}_i(x) = y_i(x)$ if $x \in X_i$ and $\hat{y}_i(x) = 0$ if $x \notin X_i$. The probabilities are constructed from the compound lottery, with probability α giving the right to play y_1 and with probability $1-\alpha$, the right to play y_2.

Convex linear operations on elements of Y^S are defined pointwise:

$$h = \alpha f + (1-\alpha) g \;\Leftrightarrow\; \forall s \in S : h(s) = \alpha f(s) + (1-\alpha) g(s). \tag{38}$$

We embed Y in Y^S by viewing elements of Y as constant acts:

$$y \in Y \Rightarrow y_s \in Y^S \text{ has } y_s(s) = y, \forall s \in S. \tag{39}$$

We call the set of constant acts L_c. We assume that L is a convex set in Y^S, containing L_c. The structure of these spaces is shown in Figure 14.

[Figure 14 about here.]

We consider a decision maker who is faced with choosing between different elements of L. We assume that the decision maker has an "at least as good as" preference relation $\succeq \subset L \times L$. f, g, and h are elements in L, and we assume that the preference relation satisfies:

Axiom A.1 Weak order

$$\forall f \forall g : f \succeq g \text{ } or \text{ } g \succeq f, \qquad f \succeq g \text{ } and \text{ } g \succeq h \Rightarrow f \succeq h. \tag{40}$$

With the weak order axiom it is straight forward to define the $\succ, \prec, \preceq, \cong$ relations in the standard way. We also assume the following axioms:

Axiom A.2 Continuity

$$\forall f \forall g \forall h : f \succ g \text{ } and \text{ } g \succ h \;\Rightarrow\; \exists \alpha \in (0,1), \exists \beta \in (0,1) : \alpha f + (1-\alpha) h \succ g \text{ } and \text{ } g \succ \beta f + (1-\beta) h, \tag{41}$$

Axiom A.3 Monotonicity

$$\forall f \forall g, \forall s \in S : f(s) \succeq g(s) \Rightarrow f \succeq g, \tag{42}$$ [22]

Axiom A.4 Nondegeneracy

$$\neg(\forall f \forall g : \ f \succeq g). \tag{43}$$

The axiom that separates the von Neumann Morgenstern (v-N M) and AA models from the GS model is the independence axiom. The v-N M/AA "type" axiom is on the form

Axiom A.5 Independence (von Neumann & Morgenstern /Anscombe & Aumann)

$$\forall f \forall g \forall h, \forall \alpha \in (0,1) : f \succ g \ \Leftrightarrow \ \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h. \tag{44}$$

The difference between the v-N M and the AA approach is the choice of L. In the v-N M approach, $L = L_c$, i.e., there is no horse dimension, and all probabilities are given. In the AA approach, L is larger than L_c, and decision makers have room to form their own subjective probabilities over the horse dimension.

The following subjective utility maximization theorem follows from (and implies) the axioms:

Theorem A.1 Subjective expected utility theorem (Anscombe & Aumannn)

Assume that a decision maker's preference relation follows axioms A.1 - A.5. Then, there exists a function, $u : \mathbb{R} \to \mathbb{R}$ (the utility function) and a probability distribution, $\mu : 2^S \to [0,1]$ (the subjective probability) such that:

$$f \succeq g \ \Leftrightarrow \ \int \int u(x) df(\mu) d\mu \geq \int \int u(x) dg(\mu) d\mu, \tag{45}$$ [23]

where df, dg are the probability distributions induced by the known probabilities of the roulette lottery associated with f and g respectively. The utility function is unique up to an affine transformation.

We call an individual who satisfies axioms A.1 - A.5 an SEU-optimizer.

For the special case when $L = L_c$, i.e., when there are only objective probabilities, the theorem reduces to the von-Neumann Morgenstern version:

Theorem A.2 Expected utility theorem (von-Neumann Morgenstern)

Assume that a decision maker's preference relation follows axioms A.1 - A.5, and $L = L_c$. Then, there exists a utility function $u : \mathbb{R} \to \mathbb{R}$, such that:

$$f \succeq g \ \Leftrightarrow \ \int u(x) df \geq \int u(x) dg, \tag{46}$$

where df, dg are the probability distributions induced by the known probabilities of the roulette lottery associated with f and g respectively. The utility function is unique up to an affine transformation.

Thus, the v-N M expected utility theorem is on the form: *If a decision maker satisfies the stated axioms and a probability is given, then he will make decisions between different lotteries as if he has a utility function, for which he maximizes expected utility.* The expected utility theorem of AA on the other hand endogenizes the probabilities, and is on the form: *If a decision maker satisfies the stated axioms, then he will make decisions between different lotteries as if he has a (subjective) probability distribution, and utility function for which he maximizes expected utility.*

The contribution of the GS approach is to weaken the independence axiom. There are two methods of doing this. We follow the multiple priors approach (Gilboa and Schmeidler 1989). The other method leads to the Choquet measure model, (Schmeidler 1989). The multiple priors approach replaces the independence axiom with:

Axiom A.6 C-independence

$$\forall f \in L, \forall g \in L, \forall h \in L_c, \forall \alpha \in (0,1) : f \succ g \ \Leftrightarrow \ \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h. \tag{47}$$

A notion of uncertainty aversion is also needed:

Axiom A.7 Uncertainty aversion

$$\forall f \forall g, \forall \alpha \in (0,1) : f \cong g \Rightarrow \alpha f + (1-\alpha) g \succeq f. \tag{48}$$

Thus, with C-independence, the decision maker is only assumed to be able to make the independence comparison with *roulette lotteries*.

With these tools, and expected utility theorems can be derived:

Theorem A.3 Multiple priors Expected Utility (MEU) Theorem (Gilboa & Schmeidler) *Assume that a decision maker's preference relation follows axioms A.1 - A.4, A.6, A.7. Then there is a closed, convex (nonempty) set of probability distributions, C, which we call the core, and a utility function, u, such that*

$$f \succeq g \ \Leftrightarrow \ \min_{\mu \in C} \int \int u(x) df(s) d\mu \geq \min_{\mu \in C} \int \int u(x) dg(s) d\mu. \tag{49}$$

Here, $df(s)$ and $dg(s)$ are the (discrete additive) probability measures induced by the roulette lotteries played if the state of the world turns out to be s, and the act chosen is f and g respectively. The utility function is unique up to an affine transformation.

We call decision makers that satisfy axioms A.1 - A.4, A.6, A.7 *MEU maximizing*, and define

$$U(f|C) \stackrel{\text{def}}{=} \min_{\mu \in C} \int \int u(x) df(s) d\mu. \tag{50}$$

B Two period investment example - Details of Section 2

B.1 Decision maker's optimization problem

The investor maximizes:

$$\begin{aligned} U(P, C_\xi) &\stackrel{\text{def}}{=} \max_\alpha \min_{\mu \in C_\xi} \int \int \log(1 + \alpha(r/P - 1)) dp d\mu \\ &= \max_\alpha \min_z \Big\{ (19/40 - z/2) \log(1 + \alpha(9/(10P) - 1)) + \\ &\quad (19/40 - z/2) \log(1 + \alpha(13/(10P) - 1)) + (1/20 + z) \log(1 + \alpha(2/(10P) - 1)) \Big\}, \\ &\qquad -\xi \times 19/20 \leq z \leq \xi/20, \end{aligned} \tag{51}$$

with respect to α. The minimum will be reached at one of the endpoints of C_ξ (or both), so we need to consider which endpoint is the right one. The first order condition is:

$$0 = \frac{(-1 + 9/(10P))(19/40 - z/2)}{1 + \alpha(-1 + 9/(10P))} + \frac{(-1 + 13/(10P))(19/40 - z/2)}{1 + \alpha(-1 + 13/(10P))} + \frac{(-1 + 1/(5P))(1/20 + z)}{1 + \alpha(-1 + 1/(5P))}, \tag{52}$$

and the solutions are:

$$\begin{aligned} \tilde{\alpha} = \Big(& 2685P - 6910P^2 + 4000P^3 - 1900Pz + 1800P^2 z \pm \sqrt{5} P \\ & \times \sqrt{651861 - 1237180P + 591220P^2 - 1366680z + 2548000Pz - 1103200P^2 z + 722000z^2 - 1368000Pz^2 + 648000P^2 z^2} \Big) \\ & / (2(-468 + 3220P - 4800P^2 + 2000P^3)). \end{aligned} \tag{53}$$

By checking the second order conditions we arrive at the positive root being a global maximum. The denominator has roots at $P = 1/5$, $P = 9/10$ and $P = 13/10$. For $P < 1/5$ or $P > 13/10$ there will be arbitrage opportunities. Inside of this region, there will be no arbitrage opportunities. We need to consider $P = 9/10$, separately, as there is a pole at this point[24].

B.2 Cost of capital increases

For the analysis of demand as a function of uncertainty in Figure 6, we fix P in (53), and calculate demand as a function of ξ for $P = 1$, 9/10 and 8/10:

$$\tilde{\alpha} = \begin{cases} (225 + 100\xi - \sqrt{5}\sqrt{5901 + 78120\xi + 2000\xi^2})/96, & P = 1, \xi \leq 11/171, \\ 9(31 - 180\xi)/(14(21 + 20\xi)), & P = 9/10, \xi \leq 31/171, \\ (-283 - 460\xi + \sqrt{202489 - 171640\xi + 211600\xi^2})/150, & P = 8/10, \xi \leq 51/171, \\ 0 & \text{Otherwise.} \end{cases} \tag{54}$$

For the preferred sets in Figure 7, we calculate

$$\begin{aligned} U(\alpha_1, \alpha_2 | C_\xi) &\overset{\text{def}}{=} \min_{\mu \in C_\xi} \int\int \log(1 + (r/P - 1)) dp_{\alpha_1,\alpha_2} d\mu \\ &= \min_{-\xi \times 19/20 \leq z \leq \xi/20} \Big\{ (19/40 - z/2) \log(1 + 9\alpha_1/10 - \alpha_1 - \alpha_2) + \\ &\qquad (19/40 - z/2) \log(1 + 13\alpha_1/10 - \alpha_1 - \alpha_2) + (1/20 + z) \log(1 + 2p_2/10 - \alpha_1 - \alpha_2) \Big\}. \end{aligned} \tag{55}$$

Once again, we only need to focus on the endpoints. We plot the result for $\xi = 0$, $\xi = 2/10$ and $\xi = 6/10$. For the preferred set with changed probabilities in Figure 15, we simply choose $\xi = 0$, $\mathbb{P}(s_{\mathrm{H}}) = 0.76$.

To see how increased uncertainty is different from changed (subjective) probabilities, we study Figure 15, where the decision maker's behavior in both situations is shown. We see that in the latter case, changing from $\mathbb{P}(s_{\mathrm{H}}) = 0.95$ to $\mathbb{P}(s_{\mathrm{H}}) = 0.76$ expands the set of preferred projects "upwards", contrary to the first situation, with increased uncertainty, which "decreases in all directions".

[Figure 15 about here.]

Finally, for the IRR calculation, Figure 8, we calculate

$$\begin{aligned} IRR(\xi) \overset{\text{def}}{=} \min_{\substack{\alpha_1+\alpha_2=1, \\ U(\alpha_1,\alpha_2|C_\xi)\geq 0}} \min_{-\xi \times 19/20 \leq z \leq \xi/20} &\Big\{ (19/40 - z/2)(1 + 9\alpha_1/10 - \alpha_1 - \alpha_2) + \\ &(19/40 - z/2)(1 + 13\alpha_1/10 - \alpha_1 - \alpha_2) + (1/20 + z)(1 + 2\alpha_2/10 - \alpha_1 - \alpha_2) \Big\} - 1, \end{aligned} \tag{56}$$

which is the formula for the internal rate of return in this two period model.

B.3 Decision maker behaves differently

For the case without uncertainty, $z = 0$, and the demand curve simply is

$$\tilde{\alpha} = \frac{2685P - 6910P^2 + 4000P^3 + \sqrt{5}P\sqrt{651861 - 1237180P + 591220P^2}}{2(-468 + 3220P - 4800P^2 + 2000P^3)}. \tag{57}$$

For $\xi = 1/20$, we have to check $z \in \{-19/200, 1/200\}$. However, in the region where utility is negative for both endpoints, the investor will be better off by simply choosing to invest everything in the riskfree project. Taking this into consideration, the resulting demand curve is:

$$\alpha(P) = \begin{cases} \dfrac{5009P - 13478P^2 + 8000P^3 + P\sqrt{10570849 - 20149324P + 9845284P^2}}{8(-234 + 1610P - 2400P^2 + 1000P^3)}, & P < 1931/2000, \\ 0, & 1931/2000 \leq P \leq 1931/2000, \\ \dfrac{5389P - 13838P^2 + 8000P^3 + P\sqrt{11(1197659 - 2272644P + 1085004P^2)}}{8(-234 + 1610P - 2400P^2 + 1000P^3)}, & P > 2119/2000. \end{cases} \tag{58}$$

Similarly for $\xi = 1/10$, we make the same considerations, ending up with:

$$\alpha(P) = \begin{cases} \dfrac{581P - 1642P^2 + 1000P^3 + P\sqrt{130705 - 250764P + 126564P^2}}{2(-117 + 805P - 1200P^2 + 500P^3)}, & P < 221/250, \\ 0, & 221/250 \leq P \leq 133/125, \\ \dfrac{2(169P - 433P^2 + 250P^3 + 2P\sqrt{3250 - 6166P + 2941P^2})}{-117 + 805P - 1200P^2 + 500P^3}, & P > 133/125. \end{cases} \tag{59}$$

C The multiperiod MEU investment model with proofs - Details of Section 3

C.1 The multiperiod MEU model

There are T time periods, where T can be very large but is finite. In each time period, there is a state space, S_i, $i = 1, \dots, T$. The total state space is

$$\mathcal{S} \stackrel{\text{def}}{=} S_1 \times S_2 \times \cdots \times S_T. \tag{60}$$

The cardinality of each S_i can vary with i, but is always finite. The total cardinality, $|\mathcal{S}| = \prod_1^T |S_i|$ is denoted with N. We always assume that $N \geq 2$.

We study a decision maker's problem of at time 0 choosing between *projects*, which generate (possibly zero) cash flows between time period 1 and T. There is a riskfree project, p_{riskfree} and (w.l.o.g.) we normalize the problem to study excess cash flow, i.e., p_{riskfree} generates zero cash flow in each time period, independently of the realization of $\mathcal{S}$.[25]

We call the intertemporal set of roulette lotteries, $\mathcal{Y} \stackrel{\text{def}}{=} Y^T$, and an intertemporal project is then a function:

$$p : \mathcal{S} \to \mathcal{Y}. \tag{61}$$

We define convex combinations of projects, $\alpha p_1 + (1 - \alpha) p_2$ in the same way as for the one-period model. We assume a convex set of projects, $\tilde{\mathcal{P}} \subset \mathcal{Y}^{\mathcal{S}}$ that contains the constant acts,

$$\tilde{\mathcal{P}}_c \stackrel{\text{def}}{=} \{p : \exists y \in \mathcal{Y}, \forall s \in \mathcal{S} : p(s) = y\}. \tag{62}$$

Following the one-period model, we assume that the decision maker has a preference relation $\succeq \subset \tilde{\mathcal{P}} \times \tilde{\mathcal{P}}$, which satisfies axioms A.1-A.4, A.6, A.7 of appendix A. The MEU theorem from appendix A follows immediately. The only difference from appendix A is that the space of outcomes X needs to be replaced by $\mathcal{X} \stackrel{\text{def}}{=} X_1 \times X_2 \times \cdots \times X_T$, but the only requirement in Gilboa and Schmeidler (1989) is that $\mathcal{X}$ is a compact set, which holds true as long as each X_i is compact, so the theorem goes through. Thus, the decision maker's MEU is

$$U(p|\mathcal{C}) \stackrel{\text{def}}{=} \min_{\mu \in \mathcal{C}} \sum_{s \in \mathcal{S}} \mu(s) \times \int \mathcal{U}(x) dp(s),^{26} \tag{63}$$

for some closed convex core of probabilities, $\mathcal{C}$ over $\mathcal{S}$ and some utility function, $\mathcal{U} : \mathcal{X} \to \mathbb{R}$. Faced with a set of projects, $\mathcal{P} \subset \tilde{\mathcal{P}}$, the decision maker solves the following optimization problem:

$$\max_{p \in \mathcal{P}} U(p|\mathcal{C}), \tag{64}$$

For elements $\mu \in \mathcal{C}$ we write μ_s for the probability that s occurs (instead of $\mu(s)$).

To make the model operational we make the following assumption on the utility function

Assumption C.1 Time-separable utility over risk dimension

$$\exists u : \cup_{i=1}^T X_i \to \mathbb{R} : \mathcal{U}(\mathcal{X}) = \sum_{i=1}^T \rho^{-i} u(x_i), \text{ for some } \rho > 0. \tag{65}$$

Without loss of generality, we can assume $u(0) = 0$. We furthermore make the standard assumption that u is the restriction to $\cup_{i=1}^T X_i$ of a weakly concave function, $u : \mathbb{R} \to \mathbb{R}$, and standard assumptions on u's asymptotic behavior. To summarize, we assume:

Assumption C.2

1. *$u(0) = 0$,*
2. *u is weakly concave and strictly increasing,*
3. *$\lim_{x \to \infty} u(x)/x = 0$,*[27]

For the propositions in Section 3, we will also need the stronger assumption on u,

Assumption C.3

u is strictly concave and two times continuously differentiable.

Moreover, for Lemma C.1 we need an assumption on the behavior of u at $-\infty$.

Assumption C.4

$\exists c : \lim_{x \to -\infty} u(x)/x < c$.

These assumptions on u only depend on the decision maker's behavior over $\tilde{\mathcal{P}}_c$, so they are neither weaker, nor stronger than when made within the v-N M framework.

C.2 Ordering of uncertainty and uncertainty aversion

The definitions in this section are dependent on the information sets that decision makers use to make their assessments. This is natural, as we now separate uncertainty, which depends on the information set, from uncertainty aversion, which is an individual quality. We will therefore write $\mathcal{C}(\mathcal{I})$ to denote the core for a decision maker, given the information set, $\mathcal{I}$.

Definition C.1 *Consider two MEU optimizing decision makers, A and B: Given the same information set $\mathcal{I}$, A is said to be weakly more uncertainty averse than B if B's core is not bigger than A's: $\mathcal{C}_{\mathrm{B}} \subseteq \mathcal{C}_{\mathrm{A}}$.*

For the definition of strict uncertainty aversion, we need a metric over the core. We use the Euclidean metric on $\mathbb{R}^N$ restricted to $\mathcal{S}$, and call it d. We also write $\partial\mathcal{C}$ for the boundary of $\mathcal{C}$ (under the d metric),

Definition C.2 *Consider two MEU optimizing decision makers, A and B: Given the same information set $\mathcal{I}$, A is said to be strictly more uncertainty averse than B, if $d(\mathcal{C}_{\mathrm{B}}, \partial\mathcal{C}_{\mathrm{A}}) > 0$.*

In words, the definitions capture that if the two decision makers are given the same information, then A will optimize over a larger core than B. We call decision makers with cores consisting of a single probability measure *uncertainty neutral* (under a certain information set). Here, we have used the definition of distances between sets

$$d(C_1, C_2) \stackrel{\text{def}}{=} \inf_{\substack{c_1 \in C_1 \\ c_2 \in C_2}} d(c_1, c_2), \tag{66}$$

where d on the r.h.s. is a pointwise distance function.

For the definition of more uncertain information, we use the following:

Definition C.3 *Consider two information sets, $\mathcal{I}_1$ and $\mathcal{I}_2$: Given a set of MEU-optimizing decision makers, $\mathcal{I}_2$ is said to be weakly more uncertain than $\mathcal{I}_1$, if for any decision maker, $\mathcal{C}(\mathcal{I}_1) \subseteq \mathcal{C}(\mathcal{I}_2)$:*

Definition C.4 *Consider two information sets, $\mathcal{I}_1$ and $\mathcal{I}_2$: Given a set of MEU-optimizing decision makers, $\mathcal{I}_2$ is said to be strictly more uncertain than $\mathcal{I}_1$, if it is weakly more uncertain, and $d(\mathcal{C}(\mathcal{I}_2), \partial\mathcal{C}(\mathcal{I}_1)) > 0$.*

In words, the definitions capture that any decision maker will choose a larger core when given information $\mathcal{I}_2$ compared with $\mathcal{I}_1$. We note that, as with risk aversion, these orderings are highly partial.

Finally, some of our propositions will depend on there being an objective measure of the probability measure an uncertainty neutral decision maker would choose. We say that this is the case if:

Assumption C.5 Objective uncertainty neutrality

There exists a set-to-point mapping:

$$\mathcal{T}(\mathcal{C}) \in \mathcal{C}, \tag{67}$$

such that, if $\mathcal{C}$ has non-empty interior, $T(\mathcal{C})$ lies in the interior of $\mathcal{C}$, and given an information set $\mathcal{I}$, and a set of decision makers, $\mathcal{DM}$:

$$\forall A \in \mathcal{DM}, \forall B \in \mathcal{DM} : \mathcal{T}(\mathcal{C}_A(\mathcal{I})) = \mathcal{T}(\mathcal{C}_B(\mathcal{I})). \tag{68}$$

We call $\mathcal{T}$ the objective mapping.

In words, this assumption means that even though each decision maker is allowed to have a unique core, everyone agrees what it means to be uncertainty neutral (compare with risk neutrality where everyone agrees on what it means to be risk neutral). The definition of uncertainty aversion agrees with GS's original, but refines it in that it makes it possible to speak of degrees of uncertainty aversion. A final note: In principle, these definitions must be made modulo a given set of decision makers. However, we do not write out this dependence explicitly, as it would become notationally cumbersome.

C.3 The sets of projects

We define sets of projects for the decision maker to choose between. There are three types of sets we are interested in. The first is the simplest:

$$\mathcal{P} = \{p_1, p_2, \ldots, p_n\} \subset \tilde{\mathcal{P}}. \tag{69}$$

We call the riskfree project generating zero excess cash flow in each time period in each state of the world, p_{riskfree}, and we assume that p_{riskfree} is not included in $\mathcal{P}$. We also assume that n is finite and, unless otherwise stated, that $n \geq 2$. The optimization problem over $\mathcal{P}$ is the simplest, and the set of optimal projects will of course always be nonempty and finite. This corresponds to picking one project, without the option to choose the riskfree project. Similarly, we can include the riskfree project in the decision problem, $\bar{\mathcal{P}} \stackrel{\text{def}}{=} \mathcal{P} \cup \{p_{\text{riskfree}}\}$. We define P^* to be the set of optimal projects, and P_0 to be the set of projects worth as least as much as the riskfree project.

We are also interested in cases when the decision maker can choose to divide investments between different projects. However, the linear combinations of acts used in Appendix A are over probabilities, and will not correspond to what we would intuitively think of as divided investments. Rather, $\alpha p_1 + (1-\alpha)p_2$ will for each of the states of the world give the decision maker the outcome of project $p_1(s)$ with probability α and the outcome of project $p_2(s)$ with probability $1-\alpha$. The definition we need is instead one where the outcome is the sum of the outcome of αp_1 and $(1-\alpha)p_2$, played independently. We therefore define a project combination operator, $\oplus : \tilde{\mathcal{P}} \times \tilde{\mathcal{P}} \to \mathcal{Y}$, where $p_1 \oplus p_2$ in each realized state of the world, s, maps to the roulette lottery with probabilities for different outcomes: $\mathbb{P}(x) = \mathbb{P}(x_1 + x_2 = x)$. Similarly, we define $\alpha \bullet p$ as the roulette lottery with the same probabilities, but with scaled outcomes, $\alpha p(s)$. We assume that the range of these operations are in $\tilde{\mathcal{P}}$. We define two extensions of the original set of projects:

Definition C.5 Set of combined projects

$$L_\Delta(\mathcal{P}) \stackrel{\text{def}}{=} \{\oplus_1^n(\alpha_i \bullet p_i) : \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0\}, \tag{70}$$

Definition C.6 Set of combined projects with short-selling

$$L_/(\mathcal{P}) \stackrel{\text{def}}{=} \{\oplus_1^n(\alpha_i \bullet p_i) : \sum_{i=1}^{n} \alpha_i = 1\}^{28}. \tag{71}$$

Each project thus comes at a normalized cost of unity when the investment is made (at time $t = 0$). We note that these operations are *not* vector space operations as in general, $p \oplus p \neq 2p$. We do the same for the set including the riskfree project, $\bar{\mathcal{P}}$ and define $L_\Delta(\bar{\mathcal{P}})$ and $L_/(\bar{\mathcal{P}})$.

We define a metric between elements in $\mathcal{Y}$ by $d(x,y) = \sum_{t=1}^{T} \int |F_y^t - F_x^t|$. Here F_y^t is the cumulative distribution function of outcomes at time t for lottery y, and similar for lottery x. We use this metric to define a metric over projects, $d_p(p_1, p_2) \stackrel{\text{def}}{=} \sum_{s\in\mathcal{S}} d(p_1(s), p_2(s))$. We note that $L_/$ (and L_Δ) are complete spaces under the d metric.

We need notation for enumerating outcomes: For a specific project, $p \in L_/$, we denote the kth outcome of the lottery, played in the specific state of the world, s, at a specific time, t, by $x_{s,k,t}$. Moreover, the conditional probability for $x_{s,k,t}$ conditional on the realized state of the world, s, is $\mathbb{P}(x_{s,k,t}|s)$. The outcomes and probabilities for the specific projects, $p_i \in \mathcal{P}$, $i = 1, \ldots, n$ are denoted by $x^i_{s,k,t}$, and $\mathbb{P}(x^i_{s,k,t}|s)$ respectively.

C.4 Existence and properties of solution to decision problem

We are interested in the properties of $P^*(L_/(\mathcal{P}))$ and $P_0(L_/(\mathcal{P}))$, which we denote by P^* and P_0 for short. We need the following lemma:

Lemma C.1 *Consider a specific set of projects, $\mathcal{P} = \{p_1, \ldots, p_n\}$, a core $\mathcal{C}$, and a utility function, u, satisfying the conditions of Assumptions C.2 and C.4. Then there exists constants $\tilde{c}$, $\tilde{c}'$, such that for all projects in $L_/(\mathcal{P})$, $p_a = \oplus_1^n(\alpha_i \bullet p_i)$ and $p_b = \oplus_1^n(\beta_i \bullet p_i)$, the following inequalities hold:*

$$|U(p_a|\mathcal{C}) - U(p_b|\mathcal{C})| \leq \tilde{c} \times d_p(p_a, p_b) \leq \tilde{c}' \|\alpha - \beta\|, \tag{72}$$

where the Euclidean metric in $\mathbb{R}^n$ is used in the second inequality.

Proof.
We begin with the left inequality. For a specific μ, we have

$$\begin{aligned} |U(p_a|\{\mu\}) - U(p_b|\{\mu\})| &= \Big|\sum_{s\in\mathcal{S}} \mu_s \Big(\int \mathcal{U}(x)\, dp_a(s) - \int \mathcal{U}(x)\, dp_b(s)\Big)\Big| \leq \\ &\leq N \max_{s\in\mathcal{S}} \Big|\sum_{t=1}^{T} \rho^{-t} \Big(\int u(x)\, dp_a^t(s) - \int u(x)\, dp_b^t(s)\Big)\Big| \leq \\ &\leq N\Big(\sum_{t=1}^{T} \rho^{-t}\Big)\Big|\sum_{t=1}^{T} \Big(\int u(x)\, dp_a^t(s) - \int u(x)\, dp_b^t(s)\Big)\Big|. \end{aligned} \tag{73}$$

For a specific s, we have

$$d(p_a(s), p_b(s)) = \sum_{t=1}^{T} \int |F^t_{p_a(s)} - F^t_{p_b(s)}|. \tag{74}$$

A consequence of Assumption C.4 is that $du/dx < c$ (where the differential has a Radon-Nikodym interpretation) implying that

$$\Big|\sum_{t=1}^{T}\Big(\int u(x)\,dp_a^t(s) - \int u(x)\,dp_b^t(s)\Big)\Big| = \Big|\sum_{t=1}^{T}\Big(\int u'(x)F^t_{p_a(s)}dx - \int u'(x)F^t_{p_b(s)}dx\Big)\Big| \leq$$

$$\sum_{t=1}^{T}\int \Big|u'(x)\Big(F^t_{p_a(s)} - F^t_{p_b(s)}\Big)\Big|dx \leq c\sum_{t=1}^{T}\int \Big|F^t_{p_a(s)} - F^t_{p_b(s)}\Big| = cd(p_a(s), p_b(s)). \tag{75}$$

Plugging in (75) into (73), we arrive at

$$|U(p_a|\{\mu\}) - U(p_b|\{\mu\})| \leq cN\Big(\sum_{t=1}^{T}\rho^{-t}\Big)d_p(p_a, p_b), \tag{76}$$

and as the constants do not depend on μ, we immediately arrive the inequality

$$|U(p_a|\mathcal{C}) - U(p_b|\mathcal{C})| \leq cN\Big(\sum_{t=1}^{T}\rho^{-t}\Big)d_p(p_a, p_b), \tag{77}$$

For the right inequality, we define

$$q = \max_{i,s,k,t} x^i_{s,k,t}, \tag{78}$$

and note that

$$d_p(p_a, p_b) = \sum_{s\in\mathcal{S}}\sum_{t=1}^{T}\int \Big|F^t_{p_a(s)} - F^t_{p_b(s)}\Big| =$$

$$\sum_{s\in\mathcal{S}}\sum_{t=1}^{T}\int\Big|\sum_k\Big(\prod_{i=1}^{n}\mathbb{P}(x^i_{s,k,t}|s)\Big)\Big(\Theta\big(x - \sum_{i=1}^{n}\alpha_i x^i_{s,k,t}\big) - \Theta\big(x - \sum_{i=1}^{n}\beta_i x^i_{s,k,t}\big)\Big)\Big|\,dx \leq$$

$$\leq \sum_s\sum_t\sum_k\sum_i q|\alpha_i - \beta_i| = \tilde{c}_0\sum_i|\alpha_i - \beta_i|. \tag{79}$$

Here, $\Theta(x)$ is the Heaviside step function. The desired result follows from the equivalence of absolute (l^1) and Euclidean distances in finite dimensions.

□

An immediate consequence is that $U(\cdot|\mathcal{C})$ is a continuous mapping and thus $P_0 = U^{-1}([0,\infty)|\mathcal{C})$, is closed. The same argument holds for $L_/(\bar{\mathcal{P}})$, $L_\Delta(\mathcal{P})$, and $L_\Delta(\bar{\mathcal{P}})$). Also, the (strict) concavity of u of implies that $P_0(L_/(\mathcal{P})$ is (strictly) concave, and the same argument holds for each combination of P^*, P_0, $L_/$ and L_Δ, $\mathcal{P}$ and $\bar{\mathcal{P}}$. Thus, all sets are (strictly) convex.

We need a no arbitrage condition:

Assumption C.6 No arbitrage

$$\forall p \in L_/(\bar{\mathcal{P}}) : \forall s \in S, \forall k : y(s)_k \geq 0 \Rightarrow p = p_{\text{riskfree}}. \tag{80}$$

This leads to the following lemma, which ensures that the preferred sets are bounded:

Lemma C.2 *If u satisfies the conditions of Assumption C.2, and the set of projects satisfies the no arbitrage condition C.6, then $P_0(L_/(\mathcal{P}))$ is bounded, i.e.,*

$$\exists c, \forall p = \oplus_1^n(\alpha_i \bullet p_i) : p \in P_0 \Rightarrow \|\alpha\| \leq c. \tag{81}$$

Proof. For a fixed core, $\mathcal{C}$, we wish to find an α_0, such that if $\max_i |\alpha_i| > \alpha_0$, then $U(\oplus_1^n(\alpha_i \bullet p_i)|\mathcal{C}) < 0$. If this is the case, then the equivalence between the maximum and Euclidean distance functions in finite dimensions immediately implies the desired result.

We have

$$U(\oplus_1^n(\alpha_i \bullet p_i)|\mathcal{C}) = \min_{\mu\in\mathcal{C}} \sum_{s\in\mathcal{S}} \sum_{t=1}^{T} \sum_{k} \mu_s \Big(\prod_{i=1}^{n} \mathbb{P}(x^i_{s,k,t}|s) \Big) u\Big(\sum_{i=1}^{n} \alpha_i x^i_{s,k,t} \Big). \tag{82}$$

We define

$$\pi = \min_{\mu\in\mathcal{C}} \min_{s,t,k,i} \mu_s \Big(\prod_{i=1}^{n} \mathbb{P}(x^i_{s,k,t}|s) \Big), \tag{83}$$

$$\overline{x} = \max_{s,t,k,i} x^i_{s,t,k}, \tag{84}$$

$$\underline{x} = \max\{x^i_{s,t,k,} : x^i_{s,t,k} < 0\}. \tag{85}$$

It is clear that, $\pi > 0$, $0 < \overline{x} < \infty$, and $\underline{x} < 0$. For specific values of $\alpha_1, \ldots, \alpha_n$, we define $\alpha = \max_i |\alpha_i|$. From (82), we immediately get:

$$U(\oplus_1^n(\alpha_i \bullet p_i)|\mathcal{C}) \leq \pi u(\alpha \underline{x}) + \sum_{s,k,t,i} u(\alpha \overline{x}) \leq \pi u'(0)\underline{x}\alpha + K u(\alpha\overline{x}), \tag{86}$$

where K is independent of α. By Assumption C.2:3, we know that there is an α_0, such that $\alpha > \alpha_0$ implies that,

$$u(\alpha\overline{x}) < \frac{\pi u'(0)|\underline{x}|}{K}\alpha, \tag{87}$$

so by (86), $U(\oplus_1^n(\alpha_i \bullet p_i)|\mathcal{C}) < 0$, whenever $\alpha > \alpha_0$, and we are done.

□

The same argument holds for $P_0(L_/(\bar{\mathcal{P}}))$, and of course for each L_Δ and P^*.

In the light of Lemma C.2 it is clear that Assumption C.4 is not needed for P_0 and P^* to be closed. The assumption was only needed to take care of arbitrarily large negative αs and βs. For bounded αs and βs it will trivially hold true, and as we have now shown that all sets involved are bounded, $U(\cdot|\mathcal{C})$ is continuous on these sets, and they will therefore be closed.

The properties of the different sets are summarized in Table 3.

[Table 3 about here.]

C.5 Proofs of propositions in Section 3

C.5.1 Proof of Proposition 3.1

We begin by proving (ii): Without loss of generality, we can assume that $\alpha_1 > 0$.

We assume that the core is smooth, i.e., the boundaries of the cores are strictly convex, twice continuously differentiable manifolds. This assumption, will be relaxed later.

For notational convenience, we introduce

$$u_s(\alpha) \stackrel{\text{def}}{=} \int \sum_{j=1}^{T} \rho^{-j} u(\alpha x) dp(s), \tag{88}$$

expected utility if α is invested in the risky project and realized state of the world is u_s. We also introduce $\mathcal{C}_1 \stackrel{\text{def}}{=} \mathcal{C}(\mathcal{I}_1)$, $\mathcal{C}_2 \stackrel{\text{def}}{=} \mathcal{C}(\mathcal{I}_2)$ and

$$U_1(\alpha) \stackrel{\text{def}}{=} U(\alpha|\mathcal{C}_1) = \min_{\mu\in\mathcal{C}_1} \int\int \sum_{j=1}^{T} \rho^{-j} u(\alpha x) dp(s) d\mu, \tag{89}$$

i.e., MEU, if α is invested in the risky project under information set $\mathcal{I}_1$.

We need a normality assumption:

Assumption C.7 Normality of risky project's expected utility *There exists a bijection, $m : \mathcal{S} \leftrightarrow \{1, \ldots, N\}$, such that for all α:*

$$\forall s, s' : m(s) \leq m(s') \Leftrightarrow u_s^{'} \leq u_{s'}^{'}. \tag{90}$$

Furthermore, for all α, the inequality is strict for some states of the world:

$$\exists s, s' : u_s^{'} < u_{s'}^{'}. \tag{91}$$

For a fixed μ,

$$\int \int \sum_{j=1}^{T} \rho^{-j} u(\alpha x) dp(s) d\mu, \tag{92}$$

is strictly concave, as it is a linear combination with positive weights of concave functions, with at least one *strictly* concave function. $U_1(\alpha)$, being a the minimum over a compact set of strictly concave functions, will of course also be strictly concave. Furthermore, the noarbitrage condition together with Assumption C.2 implies that $U_1(\pm\infty) = -\infty$.

Under information set $\mathcal{I}_1$, the investor solves

$$\begin{aligned} \max_{\alpha} U_1(\alpha) &= \max_{\alpha} \min_{\mu \in \mathcal{C}_1} \int \int \sum_{t=1}^{T} \rho^{-t} u(\alpha x) dp(s) d\mu \\ &= \max_{\alpha} \min_{\mu \in \mathcal{C}_1} \sum_{s \in \mathcal{S}} \mu_s \Big(\sum_{t=1}^{T} \rho^{-t} \sum_{k} \mathbb{P}(x_{s,k,t}|s) \times u(\alpha x_{s,k,t}) \Big). \end{aligned} \tag{93}$$

This is a standard concave optimization problem, and the properties of U_1 implies that there is a unique α for which the maximum is achieved. Moreover, the smoothness and strict convexity of $\mathcal{C}_1$ imply that a unique μ realizes the maximum. We call these α^* and μ^* respectively. We can define μ^* as a function of α: $\mu^* = \mu(\alpha^*)$. We differentiate U_1 at α^*, and get

$$U_1'(\alpha^*) = \sum_{s \in S} \mu_s' u_s + \sum_{s \in S} \mu_s u_s', \tag{94}$$

where $u_s' \stackrel{\text{def}}{=} du_s/d\alpha$, and respectively for μ. As $\mu(\alpha^*)$ minimizes $\sum_{s \in S} \mu_s u_s$, the first term must sum to zero, leading us to the following necessary and sufficient first order condition for a maximum of U_1:

$$\sum_{s \in \mathcal{S}} \mu_s^* u_s'(\alpha^*) = 0. \tag{95}$$

We trivially have $\forall s \in \mathcal{S}, u_s(0) = 0$, which together with the normality assumption implies that $u_s(\alpha)$ has the same ordering as induced by the normality, i.e.,

$$\forall \alpha > 0 : m(s) \leq m(s') \Leftrightarrow u_s(\alpha) \leq u_{s'}(\alpha), \tag{96}$$

with at least one inequality strict.

We study what happens to the optimal α and μ when the increase in uncertainty is "small", i.e., when $0 < d(\partial\mathcal{C}_2, \partial\mathcal{C}_1) < \epsilon_0$ for some "small" ϵ_0. As $\mathcal{I}_2$, is strictly more uncertain than $\mathcal{I}_1$, there is an open neighborhood around μ^* on the N-simplex that belongs to $\mathcal{C}_2$. We call the (unique) realized minimizing α and μ under $\mathcal{I}_2$, μ^{**} and α^{**} respectively. We define $\Delta\alpha = \alpha^{**} - \alpha^*$. The restrictions of the core being smooth and strictly concave implies that μ is twice continuously differentiable.

We define the minimizing core under information set $\mathcal{I}_2$, with investments fixed at α^*:

$$\tilde{\mu} \stackrel{\text{def}}{=} \arg\min_{\mu \in \mathcal{C}_2} \int \int \sum_{j=1}^{T} \rho^{-j} u(\alpha^* x) dp(s) d\mu, \tag{97}$$

The Lagrangian first order condition for constrained optimization implies that for small ϵ_0,

$$\forall s \in \mathcal{S} \; : \; \tilde{\mu}_s = \mu_s^* - \epsilon \Big(u_s(\alpha^*) - \frac{1}{N} \sum_{s' \in \mathcal{S}} u_{s'}(\alpha^*) \Big) + \mathcal{O}(\epsilon^2), \tag{98}$$

for some ϵ, $0 < \epsilon < Const \times \epsilon_0$.

The first order condition at the new optimum is:

$$\sum_{s \in \mathcal{S}} \mu_s^{**} u_s'(\alpha^{**}) = 0. \tag{99}$$

By Taylor expanding (95) around μ^* and α^*, (or equivalently, by using the implicit function theorem), and plugging (98) into (99), we get

$$0 = -\sum_{s\in\mathcal{S}}\Big(\Big(u_s(\alpha^*) - \frac{1}{N}\sum_{s'\in\mathcal{S}} u_{s'}(\alpha^*)\Big)u'_s(\alpha^*)\Big)\epsilon + \Big(\sum_{s\in\mathcal{S}}\mu^*_s u''_s(\alpha^{**})\Big)\Delta\alpha \ + \ \text{h.o.t.} \stackrel{\text{def}}{=} -A\epsilon + B\Delta\alpha + \ \text{h.o.t.} \quad (100)$$

As each u_s is weakly concave and there is at least one u_s that is strictly concave, B must be negative.

We use the normality condition to show that A is positive. We rewrite

$$\sum_{s\in\mathcal{S}}\Big(\Big(u_s - \frac{1}{N}\sum_{s'\in\mathcal{S}} u_{s'}\Big)u'_s\Big) = \frac{1}{N}\Big(N\sum_{s\in\mathcal{S}} u_s u'_s - \sum_{s,s'\in\mathcal{S}} u_{s'}u'_s\Big) = \frac{1}{N}\sum_{\substack{s,s'\in\mathcal{S}\\ m(s)>m(s')}}\Big((u_s - u_{s'})(u'_s - u'_{s'})\Big). \quad (101)$$

However, the normality condition immediately implies that each of the terms $u_s - u_{s'}$, as well as each of the $u_s^{'} - u_{s'}^{'}$ terms is non-negative when $\alpha > 0$, with at least one strictly positive term. Therefore, A is positive. To satisfy (100), $\Delta\alpha$ must therefore be negative, i.e., $\alpha^{**} < \alpha^*$.

We have shown that a "small" strict increase in uncertainty implies that α strictly decreases. To show that this result holds true for "large" increases, we view a large increase as a sum of small increases. We can for example do this by parametrizing the boundary of $\mathcal{C}_1$, and linearly transform it to the boundary of $\mathcal{C}_2$ by adding an extra "scale" parameter, λ. We thus define a parametric surface, such that:

$$\begin{aligned}\partial\mathcal{C}_1 &= \{\Theta(\varphi_1,\ldots,\varphi_{N-2};\lambda=0), \varphi_1\in\Phi_1,\ldots,\varphi_{N-2}\in\Phi_{N-2}\},\\ \partial\mathcal{C}_2 &= \{\Theta(\varphi_1,\ldots,\varphi_{N-2};\lambda=1), \varphi_1\in\Phi_1,\ldots,\varphi_{N-2}\in\Phi_{N-2}\},\end{aligned} \quad (102)$$

and such that

$$\Theta(\varphi_1,\ldots,\varphi_{N-2};\lambda_2) - \Theta(\varphi_1,\ldots,\varphi_{N-2};\lambda_1) \perp \frac{\partial\Theta(\varphi_1,\ldots,\varphi_{N-2};0)}{\partial\varphi_i}, \qquad i = 1\ldots, N-2. \quad (103)$$

Now, we can view the total increase in uncertainty as a large (possibly infinite) sequence of small increases, by choosing $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_i < \lambda_{i+1} < \cdots < \lambda_I = 1$. As each increase is small, we can use the local argument above to prove that in each step, $\Delta\alpha_i < 0$. Moreover, after the steps have been carried through, we have reached the unique optimum on $\partial\mathcal{C}_2$. The total change in α is the sum of the local changes, which will be negative, and we are done[29].

For the general case, when the boundaries of the cores are not twice continuously differentiable, we use a denseness argument. For each λ_i, we take a sequence of strictly convex, two times continuously differentiable manifolds, Θ_i^j, $j = 1, 2, \ldots$, that converge to $\Theta(\cdot;\lambda_i)$ in maximum norm. For each such pair, we define $\Delta\alpha_i^j \stackrel{\text{def}}{=} \alpha^*(i, j+1) - \alpha^*(i, j)$, where $\alpha^*(i, j)$ is the optimal fraction invested in the risky project under the strictly convex core with boundary Θ_i^j.

Now, the previous argument holds for each j, for each $\Delta\alpha_i^j$. Furthermore, the convergence in maximum norm of each sequence of manifolds, Θ_i^j to $\Theta(\cdot;\lambda_i)$, together with the strict concavity of $U(\Theta_i^j)$ immediately implies that

$$\lim_{j\to\infty}\Delta\alpha_i^j = \Delta\alpha_i, \ \forall i. \quad (104)$$

Thus, if we show that α_i^j is uniformly negative as j becomes large, we are done.

In (100), A is uniformly greater than zero (as α converges and thus is bounded, and as u, u' are continuous). A similar argument shows that B is uniformly bounded. Finally, the higher order terms are uniformly of order $o(\epsilon) + o(\Delta\alpha)$ in (100). This is seen by including second order terms, and noting that even though the μ''_s terms might become large as j approaches infinity (which can happen in regions where the core is not *strictly* concave),

$$\sum_{s\in S}\mu''_s u_s, \quad (105)$$

is uniformly of order $o(\epsilon) + o(\Delta\alpha)$, as μ''_s terms can only become large in orthogonal directions to u. Therefore, $\Delta\alpha_i = \lim_{j\to\infty}\Delta\alpha_i^j < 0$, and we are through.

The proof of (i) is immediate as $U(\cdot|\mathcal{C}(\mathcal{I}))$ is a nonincreasing function of uncertainty, which follows trivially as the minimum over a larger set can not increase.

Remark: The normality assumption is a technical condition. Without it, increasing α can actually decrease uncertainty, by approaching a perfect hedge. The idea is shown in Figure 16, for a simple example with two states of the world. To the left, a situation that satisfies the normality condition is shown. An increased uncertainty moves decreases the investment in the risky project. To the right, a nonnormal situation appears, and in some regions, an increased uncertainty leads to *increased* investments.

[Figure 16 about here.]

□

C.5.2 Proof of Proposition 3.2

We use the same notation as in the proof of proposition 3.1, but with vector notation, i.e., we introduce the vector, $\alpha = (\alpha_1, \ldots, \alpha_n)^*$, as the fraction invested in projects (we use $*$ to denote transpose, to avoid confusion with time periods, T, and derivatives, $'$). We call the project arising from investing α, p_α. We define

$$u_s(\alpha) \stackrel{\text{def}}{=} \int \sum_{j=1}^{T} \rho^{-j} u(x) dp_\alpha(s), \tag{106}$$

$$U_1(\alpha) \stackrel{\text{def}}{=} U(p_\alpha|\mathcal{C}_1) = \min_{\mu \in \mathcal{C}_1} \int \int \sum_{j=1}^{T} \rho^{-j} u(x) dp_\alpha(s) d\mu. \tag{107}$$

Proof of (i): For a point $p \notin \Gamma$, there are two states s_1, s_2, for which $u_{s_1} > u_{s_2}$. Assume that μ is an element in the core that achieves the minimum of U_1. As we are dealing with compact sets, and continuous functions, such an element can always be found. Then, for $\epsilon > 0$, choosing the new probabilities:

$$\tilde{\mu}_{s_1} = \mu_{s_1} - \epsilon, \qquad \tilde{\mu}_{s_2} = \mu_{s_2} + \epsilon, \qquad \tilde{\mu}_s = \mu_s,\ s \notin \{s_1, s_2\}, \tag{108}$$

will decrease MEU. Furthermore, by the definition of strict increased uncertainty, for ϵ small enough, $\tilde{\mu} \in \mathcal{C}_2$, and we are done.

Proof of (ii): From (i), we know that $U(p|\mathcal{C}(\mathcal{I}_2))$ is negative for all $p \in \partial P_0^1$. Furthermore, $U(\cdot|\mathcal{C}(\mathcal{I}_1))$ is continuous, and as ∂P_0^1 is compact we have:

$$\sup_{p \in \partial P_0^1} U(p|\mathcal{C}(\mathcal{I}_2)) < 0. \tag{109}$$

The continuity of $U(\cdot|\mathcal{C}(\mathcal{I}_2))$ implies that each point on ∂P_0^2 is at a positive distance from ∂P_0^1. Once again, we use compactness to take the infimum of these distances, which will be realized and therefore must be strictly larger than zero.

Thus, we have:

$$d_p(\partial \mathcal{P}_0^1, \mathcal{P}_0^2) > 0. \tag{110}$$

□

C.5.3 Proof of Proposition 3.3

We use the same notation as in the proof of proposition 3.2. We also use vector notation for expected utility in different realized states of the world, $\bar{u}(\alpha) = (u_{s_1}, u_{s_2}, \ldots, u_{s_N})^*$ for some ordering of $s \in \mathcal{S}$. Thus, $\bar{u} : \mathbb{R}^n \to \mathbb{R}^N$ maps investments in projects to implied expected utility in each state of the world.

(i): We begin with proving the proposition when the moment conditions are satisfied. As P_0^2 is the minimum over a larger core, than P_0^1 it is clear that P_0^2 is weakly included in P_0^1. We need to show that the inclusion is strict.

We define the first moment of the roulette lottery for project i in state s:

$$\tau_i^s \stackrel{\text{def}}{=} \sum_k \sum_t \mathbb{P}(x_{s,k,t}^i|s) x_{s,k,t}^i \rho^{-t}, \qquad 1 \le i \le n,\ s \in \mathcal{S}, \tag{111}$$

and the moment conditions:

Definition C.7 Moment conditions
The projects in $\mathcal{P}$ are said to satisfy the moment conditions if

$$\exists i, \exists i', \exists s, \exists s' :\ \tau_i^s \tau_{i'}^{s'} \neq \tau_{i'}^s \tau_i^{s'}. \tag{112}$$

We have $\bar{u}(0) = 0$. Furthermore, the strict concavity of u implies that 0 is a boundary point of P_0^1. To see this, assume the contrary, i.e., that 0 lies in the interior of P_0^1. Then, there is an $\alpha \neq 0$, such that $U_1(\alpha) = 0$, and $U_1(-\kappa\alpha) = 0$ for some $\kappa > 0$. But, we know from the proof of Proposition 3.1 that U_1 is strictly concave, immediately implying that $U_1(0) > 0$ and we have a contradiction.

We Taylor expand $\bar{u}$ around 0 to get:

$$\bar{u}(\alpha) = c_1 M\alpha + o(\|\alpha\|). \tag{113}$$

Here, $M \in \mathbb{R}^{N\times n}$ is a matrix with elements $M_{i,j} = \tau_j^{s_i}$, and $c_1 = u'(0)$ (where u' is the derivative of u). The moment conditions imply that the rank of M is at least two. The implicit function theorem immediately implies that there is a neighborhood of 0, in which the $\Gamma_0 \stackrel{\text{def}}{=} \{\alpha : \bar{u}(\alpha) = 0\}$ is a manifold of dimension at most $N-2$. As P_0^1 has nonempty interior, we know that its boundary is a (piece-wise) manifold of dimension $N-1$, which implies that close to 0, "almost all" $p \in \partial P_0^1$ satisfy $p \notin \Gamma_0$. For any such p, it is clear from Proposition 3.2 that $p \notin P_0^2$, and thus P_0^2 is a strict subset of P_0^1.

(ii): We now prove the result when the moment conditions are *not* satisfied, but there is a project in $L_/(\bar{\mathcal{P}})$ satisfying the normality condition. We call this project p.

As P_0^1 has non-empty interior, the rank of M can not be zero, and thus it is exactly one. Therefore, there exist s and i for which $\tau_i^s \neq 0$. For such an s, we define the vector $\tau = (\tau_1^s, \ldots, \tau_n^s)^*$. The implicit function theorem ensures that in a neighborhood of 0, Γ_0 is a manifold of dimension $N-1$, to a first order approximation defined by:

$$L_\tau = \{l \in \mathbb{R}^n : \tau^* l = 0\}. \tag{114}$$

It is clear that $p \notin L_\tau$, as any $p \in L_\tau$ will violate the normality condition at $\alpha = 0$. Therefore, there is an $\epsilon \neq 0$ such that $\epsilon \bullet p \in P_0^1$. Without loss of generality, we assume that $\epsilon > 0$. We define

$$\kappa = \max_r \{r : U(r \bullet p|\mathcal{C}(\mathcal{I}_1)) \geq 0\}. \tag{115}$$

It is clear from the continuity of $U(\cdot|\mathcal{C}(\mathcal{I}_1))$ that $\kappa \bullet p \in \partial P_0^1$. Moreover, the normality condition implies that $\kappa \bullet p \notin \Gamma_0$, as u_s has the same ordering as induced on u'_s by the normality ordering. We can once again use Proposition 3.2 to ensure that $\kappa \bullet p \notin P_0^2$ and we are done.

□

Remark: It is clear that for a set of projects to fail to satisfy both the normality and moment conditions, the projects have to be close to degenerate, both compared with each other (by the moment condition) and each project across the states of the world (by the normality condition). One such set of projects that fail to satisfy both the moment and normality condition is when all projects give rise to the same roulette lottery in every state of the world. Of course, in such a situation, the level of uncertainty is irrelevant.

C.5.4 Proof of Proposition 3.4

The assumptions on the projects are:

Assumption C.8 Assumptions on projects, $p_1, \ldots, p_n$

1. *Each project has nonnegative expected payoff in each time period,* $t = 1, \ldots, T$.
2. *Each project has a positive IRR.*
3. *There are two projects that have different IRRs.*

We denote the objective probability by μ, $\mu = \mathcal{T}(\mathcal{C})$, where $\mathcal{T}$ is the objective mapping. The decision maker invests α_i in project i. We use vector notation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. As $P_0^1 \in L_\Delta$, we only need to consider nonnegative α. The αs therefore satisfy

$$\sum_{i=1}^n \alpha_i = 1, \qquad \alpha_i \geq 0. \tag{116}$$

To calculate the IRR[30], we need to study the roots of the polynomial:

$$\begin{aligned} Q(z|\alpha) &\stackrel{\text{def}}{=} -z^T + \sum_{t=1}^T \Big(\sum_{s\in S} \sum_k \mu_s \mathbb{P}(x_{s,k,t}|s) x_{s,k,t} \Big) z^{T-t} \\ &= \sum_i \alpha_i q_i(z) = -z^T + a_{T-1} z^{T-1} + a_{T-2} z^{T-2} + \cdots + a_0. \end{aligned} \tag{117}$$

Assumptions C.8:1-2 imply that all $a_0, \ldots, a_{T-1}$ are nonnegative, with at least one positive a_i, regardless of α. We can therefore use Descartes' rule to ensure that $Q(z|\alpha)$ has exactly one positive real root regardless of α. Furthermore, theory of analytic functions ensures that the root is a continuous function of α. We call this root z_0.

We now prove that the minimal z_0 in the set of preferred projects (under information set $\mathcal{I}_1$) as a function of α can not lie in the interior of P_0^1. To do this we use a special case of Lemma 3.1 in Walden (1999).

Lemma C.3 *Suppose $Q(z)$ is a polynomial of degree T, with a simple root at z_0 and that $Q'(z_0) = R$. Furthermore, suppose $\tilde{Q}(z)$ is a polynomial of degree less than T, with $\tilde{Q}(z_0) = A \neq 0$. Then $\exists c > 0, \exists \epsilon_0 > 0 : 0 < |\epsilon| < \epsilon_0 \Rightarrow Q(z) + \epsilon\tilde{Q}(z)$ has a simple root in*

$$B(z_0 - \epsilon R/A, \gamma), \qquad \gamma = c\epsilon^2. \tag{118}$$

Here, $B(z_0 - \epsilon R/A, \gamma)$ is the open ball in the complex plane, with center at $z_0 - \epsilon R/A$ and radius γ.

Proof: See Walden (1999). The proof is a straight forward application of Rouché's theorem.

To see that the minimum IRR can not lie in the interior of P_0^1, let us assume the contrary, and that the minimum is realized by α^*. The root is real and simple (implied by Descartes rule). Moreover, for any α^{**}, with $\sum_i \alpha_i^{**} = 0$, we have $Q(z|\alpha^* + \epsilon\alpha^{**}) = Q(z|\alpha^*) + \epsilon\tilde{Q}(z)$, where $\tilde{Q}(z)$ is a polynomial of degree not higher than $T - 1$.

We can choose α^{**} such that $\tilde{Q}(z_0) \neq 0$. Assuming the contrary would imply that there is a b such that $q_i(z_0) = b$, for all i, which by (117) would imply that $b = 1$, and that $q_i(z_0) = z^T$, i.e., that $IRR(p_i) = z_0 - 1$ for all i, in violation with assumption C.8:3. Thus, there are i, and j, such that $q_i(z_0) \neq q_j(z_0)$, and by choosing $\alpha_i^{**} = 1$, $\alpha_j^{**} = -1$, and $\alpha_k^{**} = 0$, $\forall k \notin \{i, j\}$, we get $\tilde{Q}(z_0) = q_i(z_0) - q_j(z_0) \neq 0$.

Thus, the conditions of Lemma C.3 are satisfied. By choosing ϵ small enough, we get $c\epsilon^2 << |\epsilon|R/A$ and by choosing such a small ϵ, we can ensure that the positive real root of $Q + \epsilon\tilde{Q}$ is strictly smaller than that of Q. We have a contradiction and, thus, the minimum must be realized on ∂P_0^1.

From Proposition 3.2 we know that P_0^2 does not contain any elements of ∂P_0^1, so z_0 does not belong to P_0^2. Furthermore, as P_0^2 is compact, the infimum over IRR's under information set $\mathcal{I}_2$ is realized (on ∂P_0^2), and thus strictly larger than under $\mathcal{I}_1$.

□

C.5.5 Proof of Proposition 3.5

(i): For this proposition, we define projects by their α representations. With this representation, P_0^1 and P_0^2 are subsets of $\mathbb{R}^n$. The idea is that when $u(x) \equiv x$, P_0^1 and P_0^2 will be cones, and that P_0^2 will have a smaller n-dimensional solid angle than P_0^1. To show this, we proceed as follows: The condition for being in the preferred set for a specific μ is:

$$\sum_s \mu_s \sum_t \sum_k \mathbb{P}(x_{s,k,t}|s) x_{s,k,t} \geq 0. \tag{119}$$

We can rewrite this in vector notation as

$$\mu^* A\alpha \geq 0, \tag{120}$$

where $A \in \mathbb{R}^{N \times n}$, $\mu \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}^n$. The strong moment conditions imply that the rank of A is at least two.

For the case with no uncertainty, when $\mathcal{C}_1$ contains a single element, assumption (C.8:2) immediately implies that $\xi_\mu \stackrel{\text{def}}{=} \mu^* A$ has only positive coefficients, $\xi_\mu \in \mathbb{R}^n_{++}$. The set of preferred project in this case is a halfspace,

$$L_\mu \stackrel{\text{def}}{=} \{l \in \mathbb{R}^n : \xi_\mu^* l \geq 0\}, \tag{121}$$

bounded by the hyperplane $\xi_\mu^\perp \stackrel{\text{def}}{=} \{l \in \mathbb{R}^n : \xi_\mu^* l = 0\}$.

For a general core, $\mathcal{C}_1$, the set of preferred projects will be the intersection

$$P_0^1 \stackrel{\text{def}}{=} \cap_{\mu \in \mathcal{C}_1} L_\mu, \tag{122}$$

which, being an intersection of closed half-spaces will be a closed cone. Furthermore, as $F(\mathcal{I}_1) > 0$, the cone is non-trivial, implying that the boundary of the cone is a (piece-wise) manifold of dimension $n - 1$. As A has rank at least two, this means that the points on the boundary of the cone that satisfy $A\alpha = 0$ form a surface of dimension at most $n-2$. Thus, "nearly all" α on the boundary satisfy $A\alpha \neq 0$.

Pick any such α. We know that $\mu^* A\alpha = 0$, implying that $A\alpha$ can not be a constant vector. Furthermore, the definition of strictly increased uncertainty implies that for any $\beta \in \mathbb{R}^n$, such that $\sum \beta_i = 0$, for ϵ small enough,

$$\mu_2 = \mu + \epsilon\beta \in \mathcal{C}_2. \tag{123}$$

Therefore, we can always find a $\mu_2 \in \mathcal{C}_2$ such that

$$\mu_2^* A\alpha = (\mu + \epsilon\beta)^* A\alpha = 0 + \epsilon\beta A\alpha < 0. \tag{124}$$

Thus, any nontrivial α belonging to ∂P_0^1 and with $A\alpha \neq 0$, does *not* belong to P_0^2. Furthermore, the continuity of A, and compactness of $\mathcal{C}_2$ implies that there is a neighborhood around such an α that does not belong to P_0^2. Therefore, P_0^2's cone must have strictly smaller solid angle than P_0^1's.

Now, it is immediate by the definition of $F(\mathcal{I})$

$$F(\mathcal{I}) \stackrel{\text{def}}{=} \lim_{\gamma\to\infty} \frac{\lambda(\gamma B \cap P_0(\mathcal{I}))}{\lambda(\gamma B \cap \mathcal{N})}, \tag{125}$$

that when taken for two cones, F converges to the fraction of their two n-dimensional solid angles, and as the solid angle of the denominator is constantly $n \times V(n)/2$, where $V(n)$ is the volume of the n-dimensional unit ball, we are done.

(ii): This is immediate, as the set of preferred projects can not increase when uncertainty increases.

□

C.5.6 Proof of Proposition 3.6

As in Proposition 3.5, we define projects by their α representations, and view P_0^1 and P_0^2 as subsets of $\mathbb{R}^n$.

(i) The relationship between $\mathcal{DM}_1$'s and $\mathcal{DM}_2$'s utility functions can be written

$$u_1(x) = u_2(x)R(x), \tag{126}$$

where $R(x)$ is a strictly decreasing continuous function, such that $R(0) = 1$. For points on Γ, we can pick any $\mu \in \mathcal{C}_2$ and get the MEU. Specifically, we choose an μ that belongs to $\mathcal{C}_1$. The condition for lying on ∂P_0^2 is:

$$\sum_s \mu_s \sum_t \sum_k \mathbb{P}(x_{s,k,t}|s) u_2(x_{s,k,t}) = 0. \tag{127}$$

As there are non-zero outcomes, and as $R(x)$ is a strictly decreasing function, we have:

$$\sum_s \mu_s \sum_t \sum_k \mathbb{P}(x_{s,k,t}|s) u_2(x_{s,k,t}) R(x_{s,k,t}) < 0, \tag{128}$$

which immediately implies that $U_1(p) < 0$ for any project on $(\Gamma \cap \partial P_0^1) \setminus \{p_{\text{riskfree}}\}$. The result follows from the continuity of U.

(ii) In a neighborhood of p_{riskfree}, the decision makers are, to a first order approximation, riskneutral. Therefore, the implicit function theorem implies that in such a neighborhood, P_0^1 and P_0^2 are, to a first order approximation, cones.

Thus, in a small neighborhood of p_{riskfree}, B, we can use a similar argument as in the proof of Proposition 4.4: The moment conditions imply that P_0^2's cone has a smaller n-dimensional solid angle than P_0^1's, so we immediately have

$$F_B(P_0^2, P_0^1) = \frac{\lambda(P_0^2 \cap B)}{\lambda(P_0^1 \cap B)} < 1, \tag{129}$$

for a small enough neighborhood, B, of the riskfree project.

□

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Notes

[1]We use the phrase "NPV rule" for both the decision rule that the sum of discounted expected cash flows should be greater than zero, and that the internal rate of return on expected cash flows should be greater than the adjusted discount rate (the hurdle rate). Without capital constraints, and disregarding the problem of multiple IRRs, the two methods give identical results for whether a project should be undertaken or not.

[2]In the survey of Sandahl and Sjögren (2003), 52.3% of the respondents use the net present value rule, which is higher than in earlier Swedish surveys: Renck (1966) - 10.7% and Yard (1987) - 37.6%.

[3]Venture capital is used as a general term to include private equity from seed stage, through early stage and expansion stage, but with focus lying on the earlier stages of financing.

[4]David Cromwell, former CEO of JP Morgan's Private Investments branch uses a somewhat lower rate, and writes: "IRRs in excess of 25% per annum [...] are sought by investors in private equity situations", in lecture notes of MBA course in venture capital and private equity investments, Yale School of Management.

[5]The story is less clear if we wish to study *realized* rates of returns for venture capital, because of the lack of data. Thomson Financial Securities Data reports yearly aggregate returns of 22.7% for US early seed venture capital investments between 1980 - 2000. Also, in Venture Economics 1985, p.69, rates of return for 29 partnerships between 1965-1984 of 26% are reported.

This is far above the S&P yearly return of 15.7% during the same time period. However, recent studies propose that the returns might be lower. As the majority of reported rates of return are based on appraisal values, i.e., are for funds that have not liquidated, they are based on the funds' (unrealized) estimations. Cochrane (2003) argues that this creates a selection bias. When taking the bias into account, returns in his sample decrease from 108% to 15%. Chen, Baierl, and Kaplan (2002) report returns of 13.4% for 148 funds that liquidated between 1960-1999. One could also argue against basing measures on only liquidated funds. If only the poorly performing funds liquidate, we get a selection bias in the opposite direction. Thus, the difficulty in finding reliable data for the venture capital industry makes it an open question how high realized rates of returns actually are.

[6] The notation $\succ$ describes the preference relation over lotteries for an investor (with $\succeq$, $\preceq$, $\prec$, $\cong$ defined respectively), whereas $\geq$, $\leq$ are reserved for numerical inequalities.

[7]For more details, see conference material from the McKinsey broadband conference in Tel Aviv, Israel, February 7, 2002.

[8]Even though Weeds (2002) in a recent paper proposes a model with two competing firms where there are real option like noncooperative equilibria.

[9]In the investment context we will call these acts *projects*.

[10]Here, $\int \cdot \, d\mu$ has a probabilistic interpretation, i.e., $\int h \, dF \stackrel{\text{def}}{=} \int h(x) \, \frac{dF}{dx} \, dx$, where $\frac{dF}{dx}$ is the Radon-Nikodym derivative of the c.d.f, F. Thus, μ is *not* a dummy variable for integration, but represents a probability distribution over the horse dimension.

[11]This is of course not a natural assumption in a real investment context. It is used to avoid having to take "boundary conditions" into account, which would complicate the theory. In a

real life situation, we would not expect any other project than possibly the riskfree project to be held in negative amounts.

[12] This is in fundamental contrast to the SEU model where the probabilities *move* when the subjective probabilities change. The result of changed probabilities, but without uncertainty are shown in Figure 15 in Appendix B.

[13]As the payoffs are risky, we calculate the IRR using expected payoffs, throughout the paper.

[14]Technically we are dividing two sets of infinite measure. To overcome this problem, we need to take limits of bounded sets in the right way. This is the price paid for having a "risk neutral" decision maker. The precise way of doing this is given in Section 3.

[15]This type of effect could be used in an aggregate framework to explain the equity premium puzzle, as suggested in Epstein and Wang (1994). Similar arguments can also be used to explain the own-equity effect as in Boyle, Uppal, and Wang (2003) and other under-diversification puzzles as argued in Mukerji and Klibanoff (2002).

[16]As we have normalized to excess return over the riskfree project, we could have $0 < \rho < 1$,

[17]For $L_{/}(\bar{\mathcal{P}})$, investing everything in the riskfree project will of course always achieve a perfect hedge. The set will therefore typically be a manifold of dimension $n+1-N$ if $N \leq n+1$ and a single point if $N > n+1$.

[18]It can be shown that Assumption 3.2:3 implies that Γ can not cover the whole of $L_{/}(\mathcal{P})$. However, it *can* be "thick" in the sense that it has nonempty interior in $L_{/}(\mathcal{P})$.

[19]It is actually possible to choose a coarser topology, induced by the distances between the cumulative distribution functions for the respective projects. For more details, see Appendix C.

[20]Increased uncertainty *aversion* is defined in analogy with increased uncertainty, in Appendix C.

[21]In the sense of Arrow and Pratt, i.e., $\mathcal{DM}_2$'s absolute risk aversion, $-u''/u'$ is pointwise strictly higher than $\mathcal{DM}_1$'s.

[22]The preference relation on Y is here induced from the preference relation on L, and the embedding of Y in L.

[23]Here and henceforth, we use a notation with two integrals, to emphasize the two dimensions of the set-up. The notation in GS papers only uses the integral over the horse dimension, and views u as an affine function on Y. As we are working with finite state spaces the integrals should be thought of in the Riemann-Stieltjes sense.

[24]This can either be done by omitting the term from the optimization, or by asymptotics, $P \to 9/10$. Either way, it is straight forward.

[25]In principle, we would need T riskfree projects to span the full riskfree pay-off space. However, we can pick one such project to represent *the* risk free project, as it is only used as reference point, against which other projects are measured. In fact, the only project we need to compare with is the risk free project that optimizes the decision maker's utility over time (in absence of risky projects). Another reference project is the annuity project that pays the same amount in every time period. The advantage of the latter is that in the case with time-separable utility, it leads to identical excess utility functions in each time period, which simplifies notation. There-

fore, we use the annuity choice of riskfree project, but note that all results hold true with the optimal project as reference point too.

[26]In the intertemporal model, we use a sum notation over the state space instead of Riemann-Stieltjes integrals, to stress the finiteness of this space.

[27]This is not the most general form. A sufficient (but less intuitive) condition is that $\lim_{x\to\infty} \frac{u(x)}{u(-x)} = -\infty$.

[28]The symbols Δ and $/$ are used to associate with the unit simplex and a subspace respectively.

[29]The "trick" could of course also have been carried to the limit, $\Delta\lambda \to 0$, leading to:

$$\Delta\alpha = \int_0^1 \frac{d\alpha}{d\lambda}\, d\lambda < 0. \tag{130}$$

.

[30]Here, IRR is calculated over expected cash flows: $IRR \stackrel{\text{def}}{=} IRR(E((p))$, as is normally done in real world investment analyses. An alternative definition would be $\overline{IRR} \stackrel{\text{def}}{=} E(IRR(p))$.

List of Figures

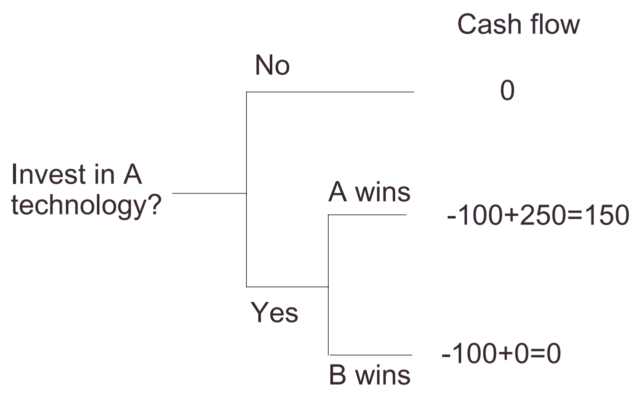

Figure 1: *Decision tree for owner in production investment example.*

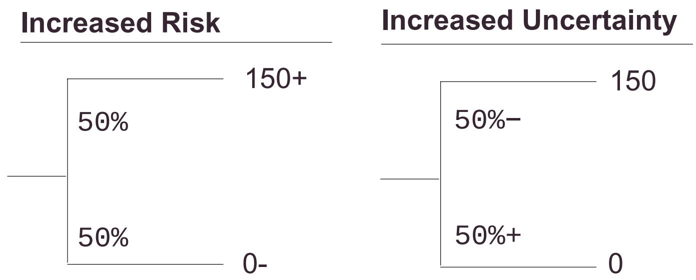

Figure 2: *Left: Increased risk enters the lottery by increasing the spread of outcomes. Right: Increased uncertainty enters the lottery by increasing the probabilities for bad outcomes.*

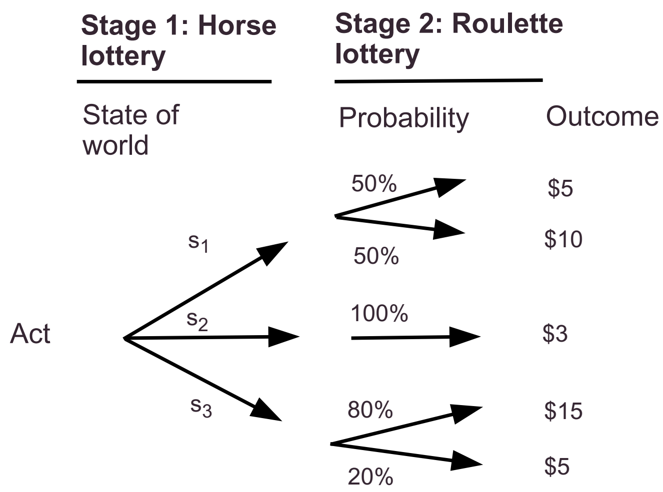

Figure 3: *Sketch of two stage lottery in Gilboa & Schmeidler model. There are three states of the world with uncertain probabilities, each giving rise to a roulette lottery with known probabilities.*

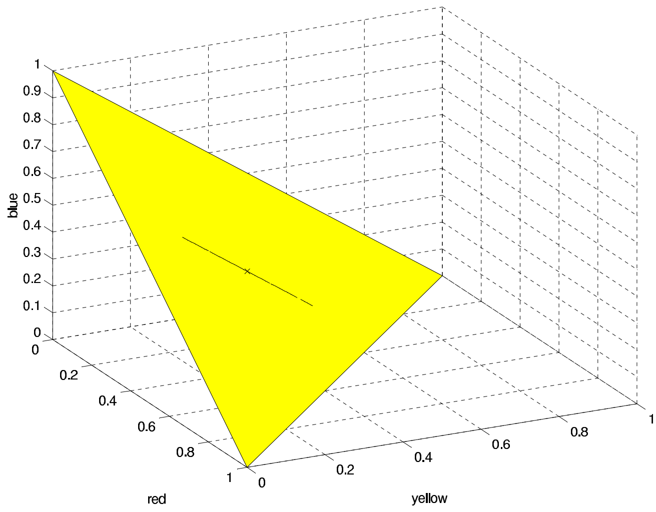

Figure 4: *Example of core of priors for decision maker that resolves Ellsberg paradox. The core of priors is the line on the unit simplex. The core is defined as:* $C = \{(\mu_{\text{red}}, \mu_{\text{blue}}, \mu_{\text{yellow}}) : \mu_{\text{red}} = 1/3, \mu_{\text{blue}} = 1/3 - \xi, \mu_{\text{yellow}} = 1/3 + \xi, \xi \in [-1/6, 1/6]\}$.

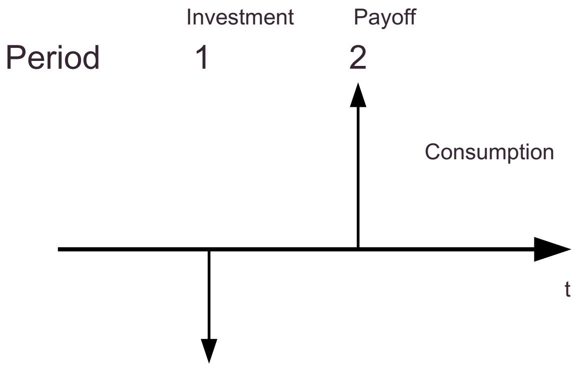

Figure 5: *Two period investment example. At $t = 1$, money is invested. At $t = 2$, payoffs are realized, and immediately consumed.*

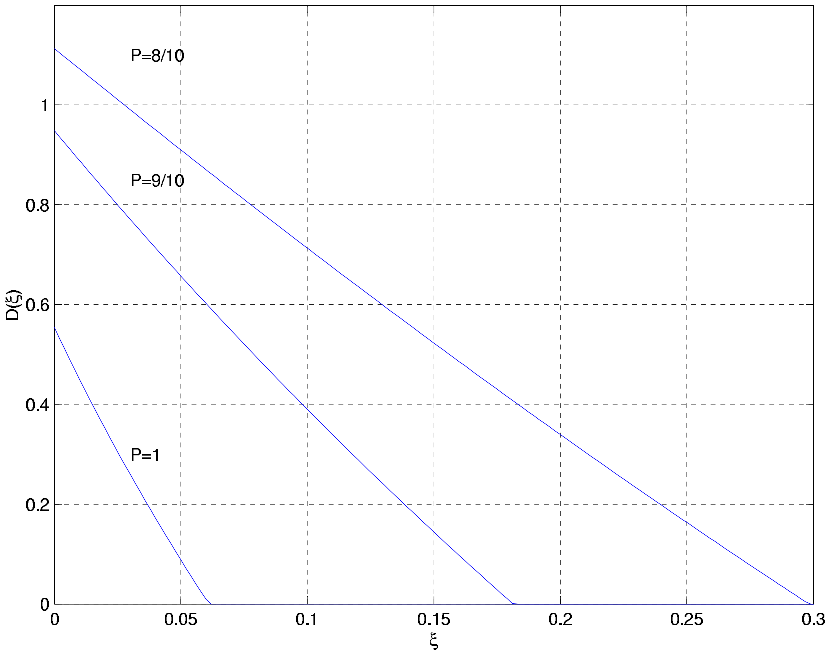

Figure 6: *Demand for risky project as a function of uncertainty for different prices:* $P = 0.8$, *0.9 and 1.0. When demand is positive, it decreases with increased uncertainty. When demand is zero, it stays zero with increased uncertainty.*

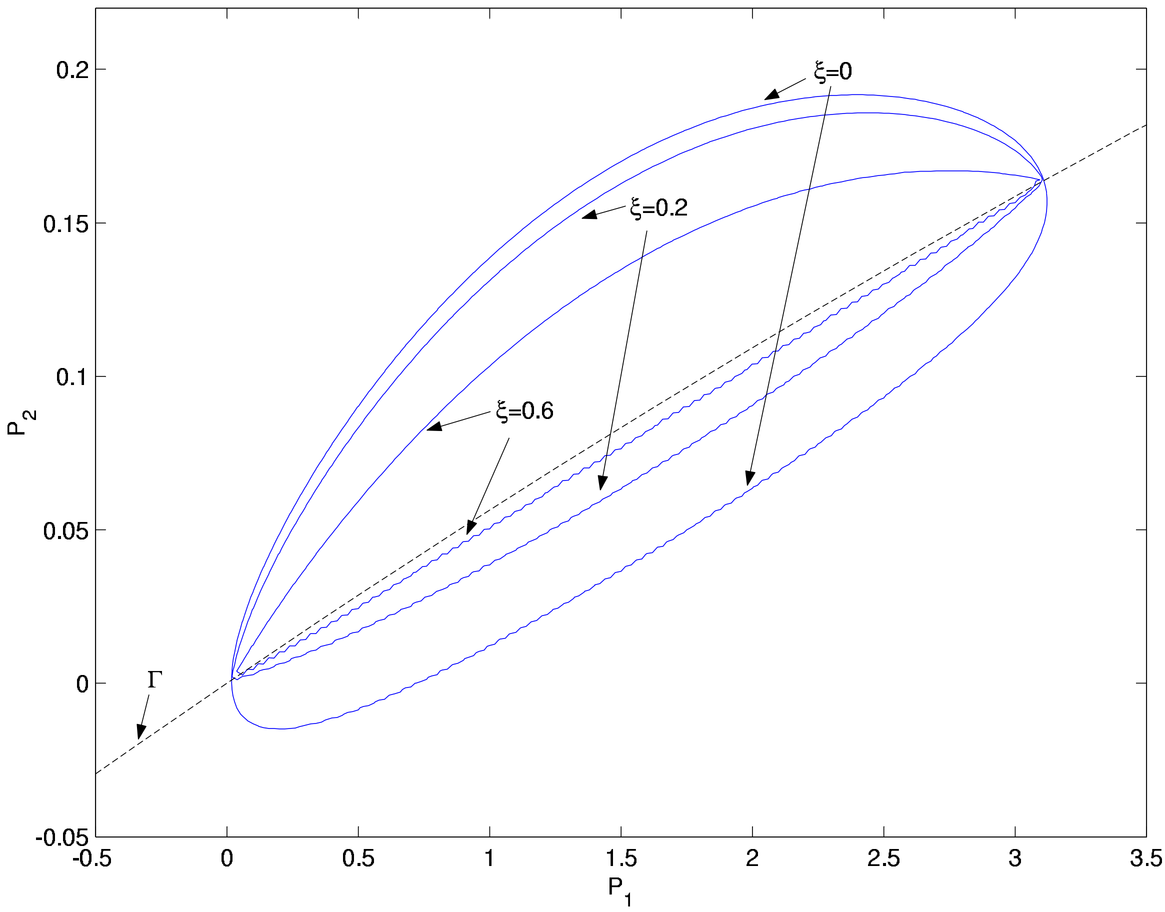

Figure 7: *Sets of preferred projects for different levels of uncertainty:* $\xi = 0$, $\xi = 0.2$, $\xi = 0.6$. *As uncertainty increases, the set of preferred projects decreases. The dotted line is perfectly hedgeable, i.e., the decision maker is indifferent towards the level of uncertainty on this line.*

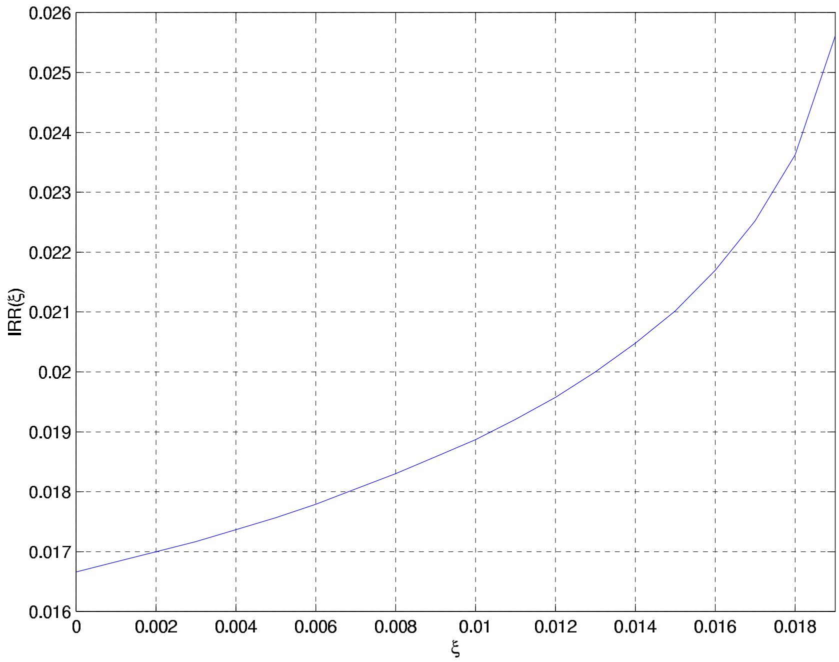

Figure 8: *Required (expected) internal rate of return to undertake a project as a function of uncertainty.*

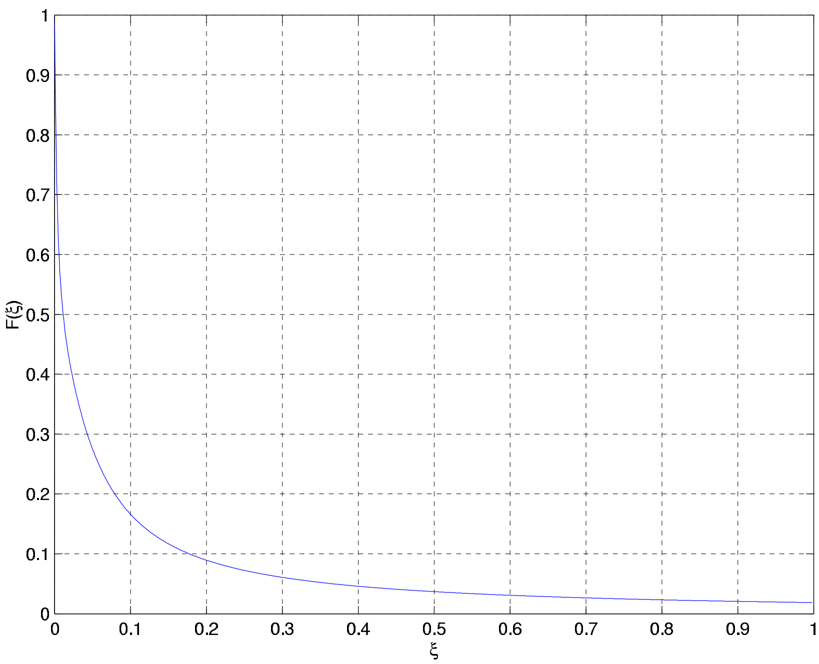

Figure 9: *Fraction of NPV positive projects that the decision maker will undertake as a function of uncertainty. For high uncertainty, a large fraction of NPV positive projects will be avoided by decision maker.*

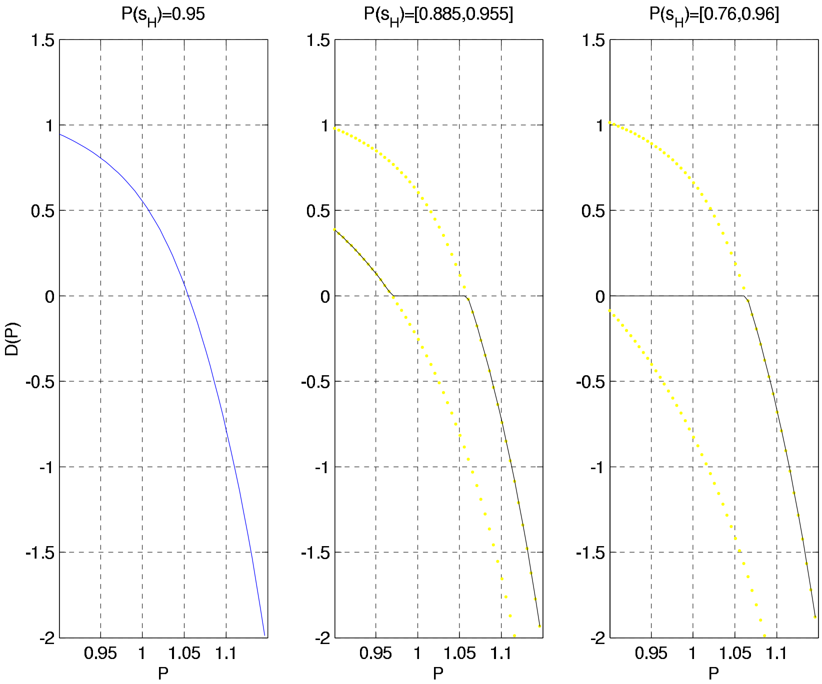

Figure 10: *Demand for risky project as a function of price for: Left - case with no uncertainty (SEU case), $\xi = 0$, Middle - $\xi = 0.2$, Right - $\xi = 0.6$. With uncertainty, the demand curve has kinks, and there is a whole region where demand is neither positive nor negative.*

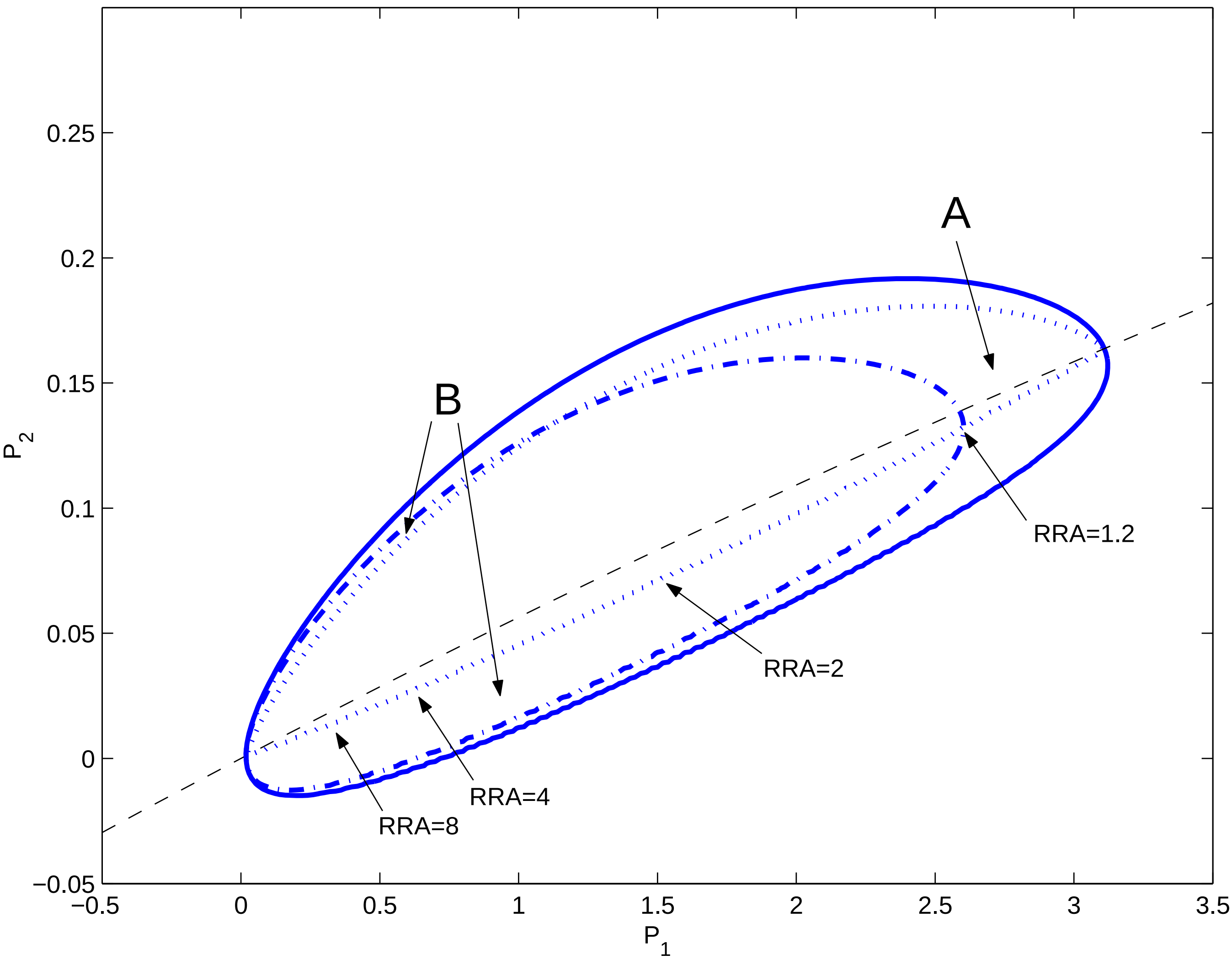

Figure 11: *Set of preferred projects under no uncertainty aversion and $RRA \equiv 1$ (solid line), under uncertainty aversion with $\xi = 0.2$ and $RRA \equiv 1$ (dotted line), and under no uncertainty and $RRA \equiv 1.2$ (dashdotted line). The A region contains projects that an MEU optimizer would invest in, but a SEU optimizer with $RRA \equiv 1.2$ would avoid. The B region contains projects that an SEU optimizer with $RRA \equiv 1.2$ would invest in, but an MEU optimizer would avoid. The implied RRA for behavior consistent with MEU optimization grows unboundedly towards the riskfree project,* $(\alpha_1 = 0, \alpha_2 = 0)$.

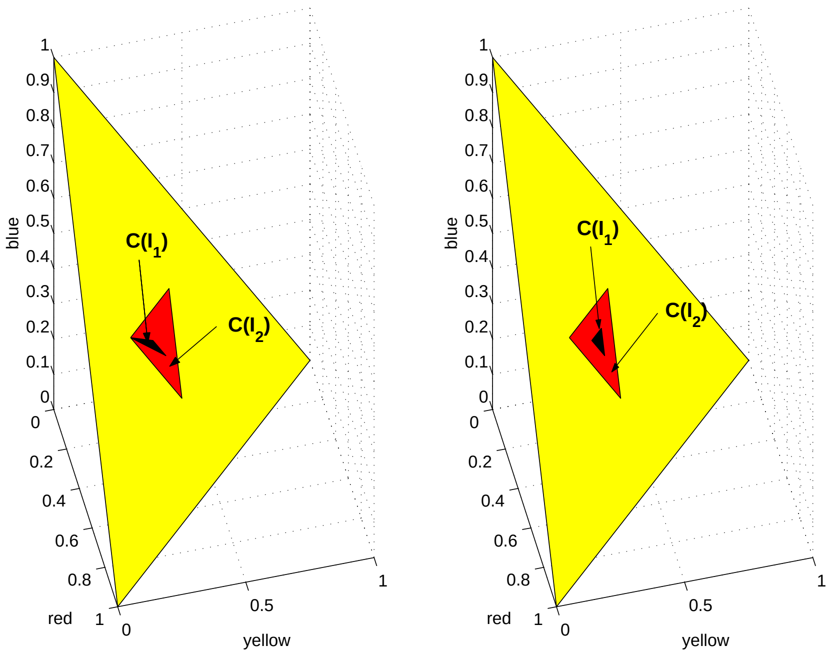

Figure 12: *Left:* $\mathcal{C}(\mathcal{I}_2))$ *is weakly more uncertain than* $\mathcal{C}(\mathcal{I}_1))$ *as the boundaries of the cores have a common point. Right:* $\mathcal{C}(\mathcal{I}_2))$ *is strictly more uncertain than* $\mathcal{C}(\mathcal{I}_1))$ *as the boundaries of the cores do not have common points.*

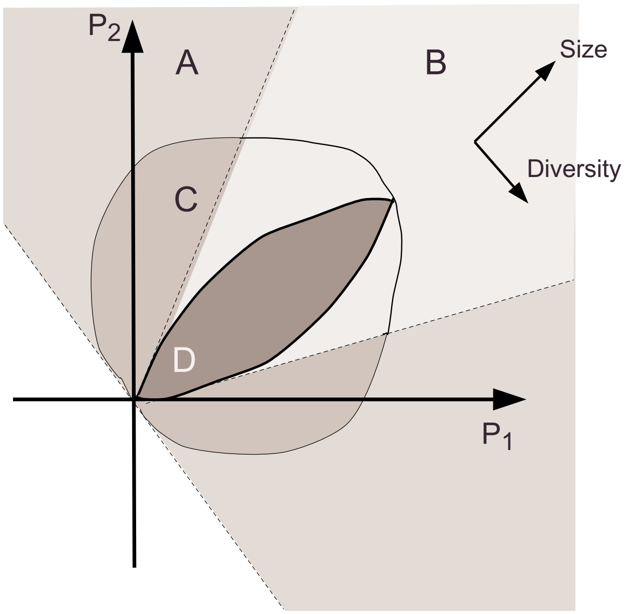

Figure 13: *Illustrative: Preferred sets for decision maker: A - Risk and uncertainty neutral, B - Risk neutral but uncertainty averse, C - Risk averse but uncertainty neutral, D - Risk and uncertainty averse.*

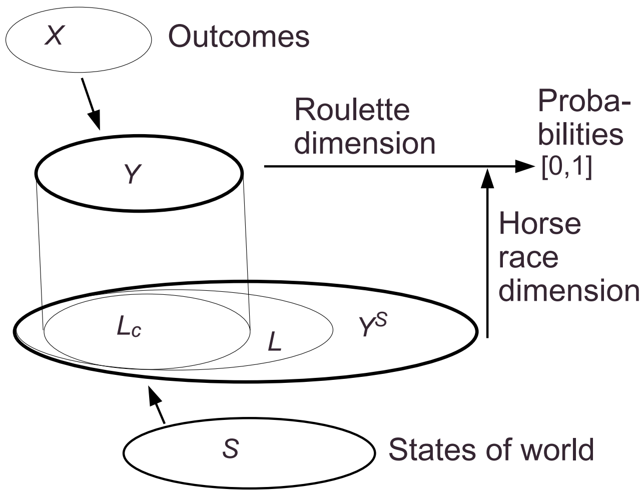

Figure 14: *Structure of spaces in Gilboa & Schmeidler/Anscombe & Aumann axiomatic approach to decision making under uncertainty.*

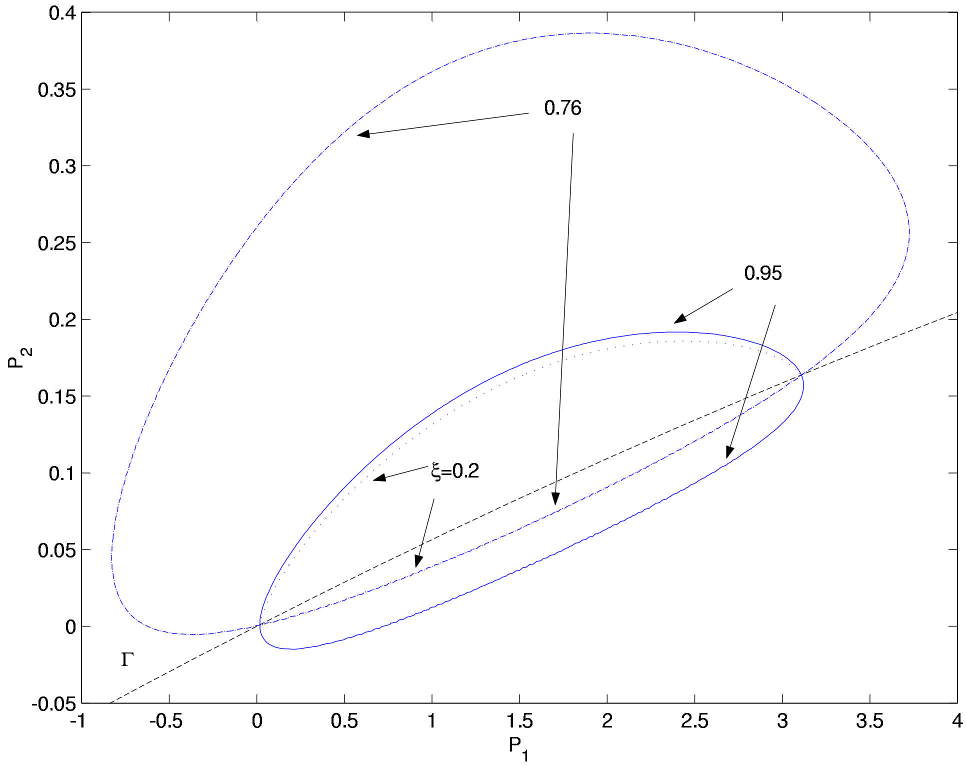

Figure 15: *Sets of preferred projects for different prior probabilities:* $\mathbb{P}(s_{\mathrm{H}}) = 0.95$ *and* $\mathbb{P}(s_{\mathrm{H}}) = 0.76$. *In contrast to the case with increased uncertainty, the set of preferred projects increases in some directions and decreases in others.*

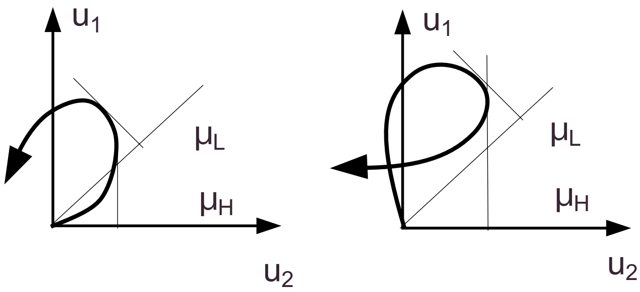

Figure 16: *Left: Decision maker's optimization problem under normality assumption. The thick line shows utility in state of the world* s_{H} *and* s_{L} *as a function of* α*, starting at* $(0,0)$ *for* $\alpha = 0$*. For low uncertainty, the tangency line perpendicular to prior* μ_1 *is the optimal solution. For high uncertainty, the tangency line perpendicular to prior* μ_2 *is optimal, implying* $\alpha_2 < \alpha_1$*. Right: Possible situation if normality assumption is violated. In this case the situation is reversed in certain regions, implying* $\alpha_2 > \alpha_1$*.*

List of Tables

$v \in V$	p_0	p_1	p_2
$(s_{\mathrm{H}}, q_{\mathrm{H}})$	1	1.3	0
$(s_{\mathrm{H}}, q_{\mathrm{L}})$	1	0.9	0
$(s_{\mathrm{L}}, q_{\mathrm{H}})$	1	0	0.2
$(s_{\mathrm{L}}, q_{\mathrm{L}})$	1	0	0.2

Table 1: Payoffs of projects in different states of the world.

$v \in V$	p_0	$p_1 + p_2$
$(s_{\mathrm{H}}, q_{\mathrm{H}})$	1	$1.3/P$
$(s_{\mathrm{H}}, q_{\mathrm{L}})$	1	$0.9/P$
$(s_{\mathrm{L}}, q_{\mathrm{H}})$	1	$0.2/P$
$(s_{\mathrm{L}}, q_{\mathrm{L}})$	1	$0.2/P$

Table 2: Returns, r, of projects in different states of the world with restricted choices, $p_1 + p_2$.

	P_0		P^*	
	$\mathcal{P}$	$\bar{\mathcal{P}}$	$\mathcal{P}$	$\bar{\mathcal{P}}$
$\mathcal{P}$	closed bounded	closed bounded nonempty	closed bounded nonempty	closed bounded nonempty
L_Δ	closed bounded convex	closed bounded convex nonempty	closed bounded convex nonempty	closed bounded convex nonempty
$L_/$	closed bounded convex	closed bounded convex nonempty	closed bounded convex nonempty	closed bounded convex nonempty

Table 3: Summary of derived results for P_0 and P^* for different combinations of $L_/$, L_Δ, and with or without riskfree project. All sets that are nonempty will consist of a single element under the additional assumption that the decision maker is strictly risk averse.