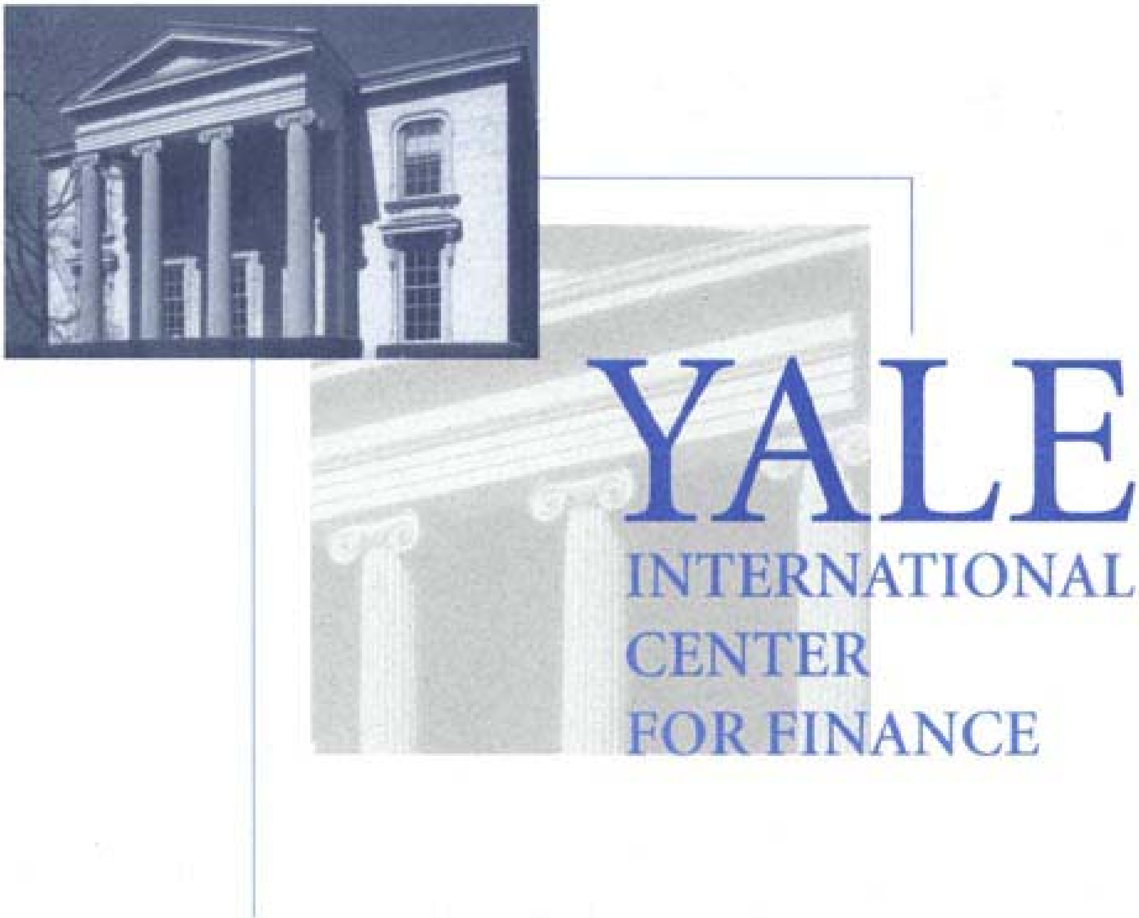

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THE SUBJECTIVE AND OBJECTIVE EVALUATION OF INCENTIVE STOCK OPTIONS

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Abstract

Incentive options are held by managers and employees who invariably hold undiversified portfolios with substantial amounts invested in their own company's common stock. This lack of diversification makes the subjective value of incentive items such as options less than their market value. This paper derives a model for the marginal value of such options or other incentive items. As such, it can be used to evaluate heterogeneous options which mature on different dates. It can also be used each time a new option is granted.

The identical model (with different parameters) can be used to determine three different values for each option, the market value, the subjective value and the objective values. The market value is the value the option would have if it were held by an unconstrained agent. The subjective value — the value of the holder — is less than the market value because the option is held in an undiversified portfolio and because it is exercised suboptimally from the market perspective. The objective value is the cost to the firm of issuing the option and lies between the market and subjective values. This value recognizes the suboptimal exercise but not the undiversified discount.

The model is no more difficult to use than is the Black-Scholes model. In fact, under the same conditions, it is simply the Black-Scholes model with modified parameters. The model can also be easily extended to handle vesting, employment termination, indexing, repricing and any number of other features found in incentive options.

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1 Introduction

Many employees, managers, and executives have undiversified portfolios with large holdings of their own firm's stock. Such portfolios are not consistent with standard finance theory which strongly recommends diversification. There are many reasons for such undiversified holdings. The stock in question may be in a pension or profit sharing plan over which the employee has no control, or it may be phantom or restricted stock or incentive options which cannot be sold. Some executives' contracts require large holdings of the company's stock. Less explicitly, the restriction may be due to a large capital gain that the manager is unwilling to realize, or the manager may simply feel "morally" constrained not to sell his company's stock.

Because managers hold undiversified portfolios, their stock ownership and equivalent items such as incentive options have a subjective value to them which is less than their market value. That is, the stock provides less utility than would an optimally balanced portfolio with the same market value.

This problem has been analyzed by others. In particular, Carpenter [1998, 2000], Hall and Murphy [2000], Kulatilaka and Marcus [1994], and Lambert, Larker, and Verrecchia [1991] have considered how a lack of diversification affect the value and incentives of option compensation. Each of these papers uses utility of terminal wealth to value a "block" of options which mature on the date that utility is evaluated, and they compute the average option value.

This paper adopts the opposite approach; it determines a model for the marginal value of an option. As such, it can be used to evaluate heterogeneous options which mature on different dates. It can also be used each time a new option is granted.

The same model (with different parameters) can be used to determine three different values for each option. The first value is the market value — the value the option would have if it were held by an unconstrained agent. The second value is the subjective value to the grantee. This value is less than the market value because the option is held in an undiversified portfolio and because it is exercised suboptimally from the market perspective. The third value is the objective value. This value recognizes the suboptimal exercise but not the undiversified discount. It is the cost to the firm of issuing the option and lies between the market and subjective values.

We examine this problem of subjective valuation in a simple framework. The investor is a manager or employee of a particular firm. He is constrained to hold at least a certain proportion of his wealth in the stock of his own firm until his retirement. After retirement his portfolio is unconstrained. The model determines the discount in subjective value which should be assigned to the option due to the manager's lack of diversification and risk aversion.

Section 2 solves the manager's consumption-investment problem. We use a standard continuous-time consumption and portfolio selection problem with a constant opportunity set. This allows us to compare our answers to those introduced by Merton [1969]. Section 3 uses the consumption-investment problem to develop subjective discounting method. Section 4 applies the subjective discounting to derivatives and other forms of incentive compensation. Section 5 extends the analysis to permit early exercise and highlights the differences between the market, subjective and objective values. Section 6 examines the effects of vesting and employment termination on subjective values. Sections 7 and 8 apply the results to illustrate the specific problems of indexed incentive options and option repricing.

2 The Constrained Portfolio Problem

In the absence of any constraints, the manager would allocate his wealth between the mean-variance efficient frontier of risky assets and default-free bonds in the usual fashion. For simplicity we assume that the continuous-time CAPM holds so the efficient portfolio is the market, but this is for the convenience of our discussion. Other equilibrium models could be used as well. Before retirement, the manager will allocate his wealth between three assets, the company's stock, the market portfolio, and default-free bonds.[1] After retirement, the manager is no longer constrained and will hold only the market portfolio and bonds until his death at time T''.[2]

We examine this problem for a manager with a power utility function defined over lifetime consumption and bequest

$$\frac{1}{\gamma}\int_0^{T''} e^{-\rho t} C_t^{\gamma}\, dt + \frac{b}{\gamma} W_{T''}^{\gamma} \tag{1}$$

[1] In general an additional asset, that portfolio (excluding the company's stock and other restricted assets like options) which is most highly correlated with his firm's stock would also be held. This portfolio would be used to reduce the undiversifiable risk imposed by the constraint of holding an excess of the company's stock. Here we assume that all the non-market risk of the individual stock is uncorrelated with other assets. The same general methodology works if this is not the case. Alternatively, we can interpret the portfolio identified as M below as the proper combination of the market and this hedging portfolio with v measuring the remaining risk that cannot be hedged away.

[2] As we shall see, the time of death and the role of bequest is unimportant in this study since the investor is unconstrained after retirement.

where b is a multiplier determining the relative contribution to utility from consumption and the bequest of wealth at time T''.

The evolution of the two risky assets is

$$\begin{aligned} dM/M &= (\mu_m - q_m)\, dt + \sigma_m\, d\omega_m \\ dS/S &= (\mu - q)\, dt + \beta\sigma_m\, d\omega_m + v\, d\omega\ . \end{aligned} \tag{2}$$

The dividend yields, q_m and q, do not affect the consumption-portfolio choice problem; they are displayed here for consistency with the pricing of derivatives developed later. The Wiener process $d\omega_m$ governs the movement of the market portfolio. The Wiener process $d\omega$ is the idiosyncratic risk of the company's stock and v^2 is the residual variance. The two Wiener processes are independent so the covariance between the stock and the market is fully captured by β. The vector of cum-dividend expected returns and the variance-covariance matrix are

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_m \\ r + \beta(\mu_m - r) \end{pmatrix} \qquad \boldsymbol{\Omega} = \begin{pmatrix} \beta^2\sigma_m^2 + v^2 & \beta\sigma_m^2 \\ \beta\sigma_m^2 & \sigma_m^2 \end{pmatrix} \tag{3}$$

where the CAPM relation $\mu = r + \beta(\mu_m - r)$ has been substituted.

The derived utility function and the optimal consumption and portfolio choices are the solution to the

$$0 = \underset{C,\mathbf{w}}{\text{Max}} \left[\frac{1}{\gamma} e^{-\rho t} C^{\gamma} + \frac{1}{2} \mathbf{w}'\boldsymbol{\Omega}\mathbf{w} W^2 J_{WW} + \left([r + \mathbf{w}'(\boldsymbol{\mu} - r\mathbf{1})] W - C \right) J_W + J_t \right] \tag{4}$$

with $J(W, T'') = e^{-\rho T''} b W^{\gamma}/\gamma$.

During the investor's retirement, the solution as given in Merton [1969] is[3]

$$w_m^* = \frac{\mu_m - r}{(1-\gamma)\sigma_m^2} \qquad w_S^* = 0 \qquad C_t^* = \Xi(t) W_t$$

where

$$\begin{aligned} \Xi(t) &\equiv A \left[\left(A b^{1/(1-\gamma)} - 1 \right) e^{A(t - T'')} + 1 \right]^{-1} \\ A &\equiv \frac{\gamma}{1-\gamma} \left[\frac{\rho}{\gamma} - r - \frac{1}{2} \frac{1}{1-\gamma} \left(\frac{\mu_m - r}{\sigma_m} \right)^2 \right]\ . \end{aligned} \tag{5}$$

[3]Note that the stock is already represented in the market portfolio. Therefore, $w_S^* = 0$ indicates that no *extra* investment is made in the stock.

The derived utility of wealth function is

$$J(W,t) = \frac{1}{\gamma} e^{-\rho t} [\Xi(t)]^{\gamma-1} W^{\gamma} \ . \tag{6}$$

Before retirement the manager is constrained to hold at least a fraction α of his wealth (beyond that represented in the market portfolio) in his company's stock.[4] We choose a proportional constraint rather than a fixed number of shares constraint to approximate its long-term intertemporal nature. The manager will usually be awarded (or receive through exercise of options) additional shares over time. Furthermore, even if he not explicitly restricted from selling some shares, he may well be subjected to implicit restraints.

For any $\alpha > 0$, the constraint will be binding and the manager's optimal portfolio should hold exactly the minimum $w_S = \alpha$. The choice between the market portfolio and the risk-free asset still must be made. The constrained maximization problem with $w_S^* = \alpha$ substituted in is

$$\begin{aligned} 0 = \operatorname*{Max}_{C,w_m} \Bigg[& \frac{1}{2}[w_m^2 \sigma_m^2 + 2 w_m \alpha \beta \sigma_m^2 + \alpha^2 (v^2 + \beta^2 \sigma_m^2)] W^2 J_{WW} \\ & + ([r + w_m(\mu_m - r) + \alpha(\mu - r)]W - C)\, J_W + J_t + \frac{1}{\gamma} e^{-\rho t} C^{\gamma} \Bigg] \end{aligned} \tag{7}$$

subject to

$$J(W,T') = \frac{1}{\gamma} e^{-\rho T'} [\Xi(T')]^{\gamma-1} W^{\gamma} \ .$$

The last condition comes from matching utility at retirement to the solution (6) of the unconstrained problem after retirement.

The first-order conditions for a maximum are

$$\begin{gathered} (w_m + \alpha\beta)\sigma_m^2 W^2 J_{WW} + (\mu_m - r) W J_W = 0 \\ e^{-\rho t} C^{\gamma-1} - J_W = 0 \end{gathered} \tag{8}$$

Solving these gives a constrained optimum of

$$\begin{gathered} w_m^* = -\frac{J_W}{W J_{WW}} \frac{\mu_m - r}{\sigma_m^2} - \alpha\beta \qquad\qquad w_S^* = \alpha \\ C^* = \left(e^{\rho t} J_W\right)^{1/(\gamma-1)} \ . \end{gathered} \tag{9}$$

[4]Let m_S be the fraction of the market portfolio which stock S represents, then the manager's holding of his own stock is $\alpha + w_m^* m_S$. Therefore, if the manager is constrained to hold at least the fraction $\overline{\alpha}$ in his own stock, we use an excess constant of $\alpha = \overline{\alpha} - w_m^* m_S$. Of course, except for very large companies m_S will be economically insignificant.

As usual we guess a solution for the derived utility function of the form $e^{-\rho t}[\xi(t)]^{\gamma-1}W^{\gamma}/\gamma$, solve for the optimal controls using (9), and substitute into (7). We can then solve for the unknown constant and the constrained optimum

$$w_m^* = \frac{\mu_m - r}{(1-\gamma)\sigma_m^2} - \alpha\beta \qquad w_S^* = \alpha \qquad C^* = \xi(t)W$$

where

$$\begin{aligned} \xi(t) &\equiv a\left[\left(\frac{a}{\Xi(T')} - 1\right)e^{a(t-T')} + 1\right]^{-1} \\ a &\equiv \frac{\gamma}{1-\gamma}\left[\frac{\rho}{\gamma} - r - \frac{1}{2}\frac{1}{1-\gamma}\left(\frac{\mu_m - r}{\sigma_m}\right)^2 + \frac{1}{2}\alpha^2 v^2(1-\gamma)\right] \\ &= A + \frac{1}{2}\gamma\alpha^2 v^2 \, . \end{aligned} \tag{10}$$

The derived utility function is

$$J(W,t) = \frac{1}{\gamma}e^{-\rho t}[\xi(t)]^{\gamma-1}W^{\gamma} \, . \tag{11}$$

The optimal holding in the market portfolio is less than the unconstrained holding by $\alpha\beta$. This serves to reduce the extra systematic risk which otherwise would be added by the forced holding of an excess of the stock. The optimal holding of bonds is

$$1 - w_m^* - \alpha = 1 - \frac{\mu_m - r}{(1-\gamma)\sigma_m^2} + \alpha(\beta - 1) \, . \tag{12}$$

This is greater (less) than the unconstrained holding if the company's stock's beta is greater (less) than one. Holding more (fewer) bonds serves to reduce (increase) the portfolio's effective leverage back closer to the desired level if beta is greater (less) than one. Optimal consumption has the same form as in the standard problem; however, the amount will differ as derived the utility function, J, will be different.

Were there no portfolio constraint on the manager, he would invest and consume as given in (5) throughout his life. From the last line in (10), $a > A$ if and only if $\gamma > 0$, and from the second line in (10), $\partial\xi(t;a)/\partial a > 0$. Therefore, compared to an unconstrained investor, the constrained manager consumes a smaller (larger) fraction of his wealth if his relative risk aversion is greater (less) than one. Similarly his marginal utility of wealth is larger (smaller) in the constrained problem if his relative risk aversion is greater (less) than one. Utility, of course, is always higher for the unconstrained problem. This dichotomy is due to income-like and substitution-like effects. The constraint makes investment for the future more costly (in terms of utility) and hence increases the relative cost of future consumption.

This causes a substitution into current consumption. However, the constraint also reduces overall utility, and this reduction in "income" reduces current consumption. As usual, if relative risk aversion is greater (less) than one, the income (substitution) effect dominates.

Prior to retirement, the evolution of wealth, net of consumption, is

$$
\begin{aligned}
dW/W &= [r + w_m^*(\mu_m - r) + \alpha(\mu - r) - \xi(t)]\ dt + w_m^*\sigma_m\, d\omega_m + \alpha v\, d\omega \\
&= \left[r + \frac{(\mu_m - r)^2}{(1-\gamma)\sigma_m^2} - \xi(t)\right] dt + \frac{\mu_m - r}{(1-\gamma)\sigma_m}\, d\omega_m + \alpha v\, d\omega\ .
\end{aligned}
\tag{13}
$$

Using Itô's lemma the evolution of the manager's marginal utility is

$$
\begin{aligned}
dJ_W &= \frac{\partial J_W}{\partial t} dt + \frac{\partial J_W}{\partial W} dW + \frac{1}{2}\frac{\partial^2 J_W}{\partial W^2} dW^2 \\
&= \left[-\rho J_W + (\gamma - 1)e^{-\rho t}[\xi(t)]^{\gamma-2}\xi'(t)\right] dt \\
&\qquad + [\xi(t)]^{\gamma-1}\left[W^{\gamma-2}\, dW + \frac{1}{2}(\gamma-2)W^{\gamma-3}\, dW^2\right] \\
\frac{dJ_W}{J_W} &= -\left[r - (1-\gamma)\alpha^2 v^2\right] dt - \frac{\mu_m - r}{\sigma_m} d\omega_m - (1-\gamma)\alpha v\, d\omega
\end{aligned}
\tag{14}
$$

before retirement and

$$
\frac{dJ_W}{J_W} = -r\, dt - \frac{\mu_m - r}{\sigma_m}\, d\omega_m \tag{15}
$$

after retirement. Note that the evolution of J_W after retirement is identical to that before retirement with $\alpha = 0$.

3 Subjective Discounting

In the absence of arbitrage, there is a martingale pricing process Θ_t which can be use to value any asset or future cash flow. The product $\Theta_t V_t$ is a martingale for the (cum-dividend) value V_t of any asset. For diffusion processes, this can be expressed as $\mathbb{E}[d(\Theta V] = 0$. For the standard portfolio problem with no constraints, one martingale pricing process is the marginal utility of any investor.

Similarly, the constrained manager can compute a *subjective value* for any asset using his own marginal utility function as a martingale pricing process, $\Theta(\alpha) = J_W$. From (14), the dynamics of the subjective martingale pricing process are

$$\frac{d\Theta(\alpha)}{\Theta(\alpha)} = \begin{cases} -\left[r - (1-\gamma)\alpha^2 v^2\right] dt \\ \qquad -\dfrac{\mu_m - r}{\sigma_m}\, d\omega_m - (1-\gamma)\alpha v\, d\omega & t < T' \\ \\ -r\, dt - \dfrac{\mu_m - r}{\sigma_m}\, d\omega_m & T' < t < T'' . \end{cases} \tag{16}$$

The after-retirement subjective pricing process is also the market martingale pricing process.

We refer to present values computed using $\Theta(\alpha)$ with $\alpha > 0$ as subjective values and present values computed using $\Theta(0)$ as objective or market values.[5] Two features of subjective pricing are immediately obvious. First, the subjective interest rate, the negative of the growth rate in $\Theta(\alpha)$, is lower than the market interest rate. Second any covariance with the residual risk of the company's stock will reduce subjective prices.

The subjective interest rate

$$\hat{r} \equiv r - (1-\gamma)\alpha^2 v^2 \tag{17}$$

is lower than the actual interest rate by an amount equal to the product of the relative risk aversion, the square of the stock-holding constraint, and the residual variance. This means that any certain payment in the future has a subjective present value to the manager higher than it's market value. The intuition for this result is immediate. One dollar at time T has a higher subjective value than e^{-rT} now because the latter would have to be invested in a suboptimal fashion.

The subjective valuation of risky cash flows is also affected. Let X and $\hat{X}$ denote the market and subjective values of a future payment. Express the subjective dynamics as

$$d\hat{X}/\hat{X} = \mu_{\hat{X}}\, dt + \beta_X \sigma_m\, d\omega_m + v_X\, d\omega + \sigma_\epsilon\, d\omega_\epsilon . \tag{18}$$

The three Wiener processes capture the unexpected changes in the value of the cashflow that are correlated with the market, correlated with the residual variation of the stock, and uncorrelated with both. $\mu_{\hat{X}}$ is the subjective rate of return (or discount rate) required to

[5] Under some conditions objective and market values will differ. For example, incentive options are often exercised suboptimally *from a market perspective*. The objective value of such an option is determined using $\Theta(0)$ and the "suboptimal" exercise policy. The market value is determined using the optimal exercise policy.

hold this asset. We can determine the subjective discount rate using the martingale pricing relation

$$\begin{aligned} 0 =& \mathbb{E}[d(\Theta\hat{X})] = \mathbb{E}[\Theta \cdot d\hat{X} + \hat{X} \cdot d\Theta + d\hat{X} \cdot d\Theta] \\ =& \Theta\hat{X}\left[\mu_{\hat{X}} - \hat{r} - \beta_X(\mu_m - r) - (1-\gamma)v_X \alpha v\right] dt \ . \end{aligned} \tag{19}$$

Since the CAPM is assumed to hold for the true discount rates, $\mu_X = r + \beta_X(\mu_m - r)$. Therefore,

$$\begin{aligned} \mu_{\hat{X}} =& \mu_X - (r - \hat{r}) + (1-\gamma)\alpha v v_X \\ =& \mu_X + (1-\gamma)\alpha v(v_X - \alpha v) \ . \end{aligned} \tag{20}$$

The subjective discount rate has two differences from the market discount rate for the asset. It is lower by the subjective interest rate differential in (17) but higher by an amount required to offset the company-specific unsystematic risk, if any, of the asset.

For the market portfolio or any portfolio with no company-specific risk ($v_X = 0$), the subjective discount rate will be lower than the objective discount rate by an amount equal to the subjective interest rate differential. As a result, subjective values of such assets will exceed market values just as they do for risk-free payments. For assets with company-specific risk, subjective discount rates can be higher or lower than their market counterparts. The more company-specific risk there is the higher will be the subjective discount rate and the lower the subjective value. If the company-specific risk of the asset is smaller (larger) than αv, the subjective discount rate will be less (greater) than the market discount rate. For the stock itself $v_X = v$ and

$$\hat{\mu} = \mu + (1-\gamma)\alpha(1-\alpha)v^2 \tag{21}$$

so the stock's subjective discount rate must exceed the market rate and the stock's subjective value must be less than the market value.

Paradoxically, the subjective discount rate is not monotone but quadratic in the constraint, α. This paradox is explained by noting that the decrease in the subjective interest rate is proportional to the square of the constraint. However, the increase in the subjective risk premium

$$\mu_{\hat{X}} - \hat{r} = \mu_X - r + (1-\gamma)\alpha v v_X \tag{22}$$

is proportional to the constraint. Therefore, when $\alpha > v_X/2v$ the subjective discount rate is decreasing in α. For assets with little correlation with the company's stock ($v_X \approx 0$), this will be true. Their subjective discount rates would be lower and their subjective values higher if the constraint were tightened. For assets with large correlations with the company's stock, this condition will typically not be met unless the constraint is severe, and such assets will be relatively less preferred if the constraint becomes tighter. This will usually be true only for assets like the stock itself or derivative contracts written on the stock.

4 The Subjective Evaluation of Compensation

Stock-based payoffs are a common component of managerial compensation. The methodology developed here allows us to determine both the subjective valuation and incentive effects of derivatives like incentive stock options and phantom or restricted stock.

4.1 Subjective Valuation

Let $F(S,t)$ denote the subjective value of the option or other compensation item. Then by Itô's lemma[6]

$$\begin{aligned}
0 =& \mathbb{E}[d(\Theta F(S,t))] \\
=& \mathbb{E}\left[\Theta F_S\, dS + \Theta F_t\, dt + \tfrac{1}{2}\Theta F_{SS} dS^2 + F\, d\Theta + d\Theta\, F_S\, dS\right] \\
=& \Theta\left[(\mu - q)SF_S + F_t + \tfrac{1}{2}\sigma^2 S^2 F_{SS} - [r - (1-\gamma)\alpha^2 v^2]F \right. \\
& \left. - [\beta(\mu_m - r) + (1-\gamma)\alpha v^2]SF_S\right] dt \ .
\end{aligned} \tag{23}$$

Using $\beta(\mu_m - r) = (\mu - r)$, we can simplify this equation to

$$\begin{aligned}
0 =& \tfrac{1}{2}\sigma^2 S^2 F_{SS} + [r - q - (1-\gamma)\alpha v^2]SF_S - [r - (1-\gamma)\alpha^2 v^2]F + F_t \\
=& \tfrac{1}{2}\sigma^2 S^2 F_{SS} + (\hat{r} - \hat{q})SF_S - \hat{r}F + F_t
\end{aligned} \tag{24}$$

$$\text{where} \qquad \hat{r} \equiv r - (1-\gamma)\alpha^2 v^2 \qquad \hat{q} \equiv q + (1-\gamma)\alpha(1-\alpha)v^2 \ .$$

This equation applies only for times before retirement at T'. After that, the usual Black-Scholes equation will apply.[7]

The partial differential equation will be recognized as the Black-Scholes equation with discounting at the subjective interest rate, $\hat{r}$, and a subjective adjustment to the dividend yield. From (21), $\hat{q} - q = \hat{\mu} - \mu$; therefore, the risk-neutral drift used in the option pricing equation is $\hat{r} - q + (\mu - \hat{\mu})$. The last term must be added to the risk-neutral drift for determining subjective prices because the state variable used is the actual stock price and

[6] Note the total risk of the stock is $\sigma^2 \equiv \beta^2\sigma_m^2 + v^2$.

[7] In the examples below, we assume that the contracts terminate before the manager's retirement, $T < T'$. Contracts which terminate later can be valued in a similar fashion with a "blended" interest rate and dividend yield.

not the subjective one. The adjustment is made because the stock price is growing at the rate μ and not $\hat{\mu}$. The subjective value of various compensation packages can now be determined from their payoffs.

As a first example, we compute the subjective value of a restricted share of stock which must be held until time T. The payoff on this contract is $F(S,T) = S$. The subjective present value of having the stock at time T with the restriction lifted is the solution to (24) with $F(S,T) = S$[8]

$$\widehat{PV}_t[S_T] = S_t e^{-\hat{q}(T-t)} \ . \tag{25}$$

The subjective present value of the share of stock includes the intervening dividends. If these may be approximated as being paid continuously at a constant yield, q, the subjective value of a share restricted until time T is

$$\hat{S}(S,t;T) = \widehat{PV}_t[S_T] + \int_t^T \widehat{PV}_t[qS_u]\,du = S\left[\frac{q}{\hat{q}} + e^{-\hat{q}(T-t)}\left(1 - \frac{q}{\hat{q}}\right)\right] \ . \tag{26}$$

The difference between the subjective value of a share of stock and the market value can be substantial. For example, consider a manager with a relative risk aversion of 5 ($\gamma = -4$) who holds 50% of his wealth in his company's stock with a dividend yield of 2% and a residual risk of $\nu = 20\%$. His subjective valuation of a share of stock restricted for 5 years is only 78.9% of the market value. For a less extreme case, $\gamma = -2$ and $\alpha = 25\%$, the subjective value is 89.9% of the market value. For a more extreme case, $\gamma = -6$ and $\alpha = 50\%$, the subjective value is only 71.8% of the market value.

[8] This present value can also be computed as

$$\widehat{PV}_t[S_T] = e^{-\hat{\mu}(T-t)}\mathbb{E}[S_T] = e^{-\hat{\mu}(T-t)} S_t e^{(\mu-q)(T-t)} = S_t e^{-\hat{q}(T-t)} \ .$$

Incentive stock options can be valued in the same way. By examination of (24), the subjective value of an incentive stock option is given by the Black-Scholes model, $\mathcal{C}(\cdot)$, with an adjusted interest rate and dividend yield[9]

$$\hat{C} = \mathcal{C}(Se^{-\hat{q}(T-t)}, T-t; X, \hat{r}, \sigma) = Se^{-\hat{q}(T-t)}\Phi\big(H(X)\big) - Xe^{-\hat{r}(T-t)}\Phi\big(h(X)\big)$$
$$\text{where} \quad H(X) \equiv \frac{\ell n(S/X) + \left(\hat{r} - \hat{q} + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad h(X) \equiv H(X) - \sigma\sqrt{T-t}\,. \tag{27}$$

Since $\hat{q} > q$ and $\hat{r} < r$, the subjective value of the option is less than its market value. The manager values the compensation in the form of stock options at less than the cost of providing it to him apart from incentive effects which may lead to increase the stock's market value.

Table 1 shows the subjective values of incentive options for some typical cases at issuance with a maturity of ten years and after one year assuming the stock price has moved by 15%. As discussed, the subjective value is smaller than the objective or market value. Therefore, standard option pricing techniques will overestimate the value that the manager will perceive in an option grant. The more risk-averse is the manager or the greater is the stock restriction in place, the smaller is the subjective value of the option. As the table indicates, the difference between the market and subjective values can be substantial. The subjective value is less than half the market value in many circumstances which should occur frequently and is less than 10% of the market value in cases which should not be exceptional.

Note that for the tighter restrictions ($\alpha \geq 50\%$) and more risk averse managers ($\gamma \leq -4$), the subjective value of the in-the-money options can be less than their intrinsic values even though the stock is not paying dividends. This result is due to the high effective dividend

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[9]Usually the stock received when an incentive options is exercised cannot be sold for a six months. So when an incentive option is exercised a "restricted" share is received. As shown in (26), a restricted share is worth less than an unrestricted share. If we take this factor into account, the condition at expiration of an incentive option is $\hat{C}(S,T) = \mathrm{Max}[RS - X, 0]$ where $R \equiv [q/\hat{q} + e^{-0.5\hat{q}}(1 - q/\hat{q})]$ is the reduction factor for a restricted share as given in (26). The value of the option is then $\mathcal{C}(RSe^{-\hat{q}(T-t)}, T-t; X, \hat{r}, \sigma)$. For typical parameters, this correction gives a reduction in the option's value of only a few percent. For example for $\alpha = 25\%, \gamma = -4, \sigma = 30\%, \upsilon = 20\%, r = 5\%, q = 0$, the market and subjective values are reduced by 3.0% and 3.5%, respectively.

Other dividend or interest rate structures can be accommodated as well. For example, for n discrete dividends at a constant yield of y and/or a sloped yield curve with zero-coupon bond prices $B(\tau)$, the subjective value of the option would be

$$\check{C} = \mathcal{C}(S(1-y)^n e^{-\check{q}(T-t)}, T-t; XB(T-t), \check{r}, \sigma)$$
$$\text{where} \qquad \check{q} \equiv (1-\gamma)\alpha(1-\alpha)\upsilon^2 \qquad \check{r} \equiv -(1-\gamma)\alpha^2\upsilon^2\,.$$

Here $\check{q}$ and $\check{r}$ are only the alteration in the dividend yield and interest rate. The actual dividend and interest effects are captured in y and $B(\cdot)$, respectively.

Table 1: **Subjective Values of Stock Options**

$X = 100 \quad r = 5\% \quad q = 0\% \quad \sigma = 30\% \quad \upsilon = 20\%$

Subjective value of call: $\hat{C}$ | **Subjective value of call as percent of market value:** $\hat{C}/C$

$S = 100, T - t = 10$: Market value of option = 52.57

α	γ = 0	−2	−4	−6	γ = 0	−2	−4	−6
10%	49.48	43.75	38.55	33.86	94.1%	83.2%	73.3%	64.4%
25%	45.81	34.26	25.07	17.91	87.1%	65.2%	47.7%	34.1%
50%	41.76	24.69	13.22	6.32	79.4%	47.0%	25.1%	12.0%
75%	39.81	19.55	7.51	2.17	75.7%	37.2%	14.3%	4.1%

$S = 85, T - t = 9$: Market value of option = 37.66

α	γ = 0	−2	−4	−6	γ = 0	−2	−4	−6
10%	35.46	31.37	27.67	24.32	94.2%	83.3%	73.5%	64.6%
25%	32.80	24.51	17.92	12.80	87.1%	65.1%	47.6%	34.0%
50%	29.73	17.39	9.25	4.42	78.9%	46.2%	24.6%	11.7%
75%	28.05	13.38	5.07	1.48	74.5%	35.5%	13.5%	3.9%

$S = 100, T - t = 9$: Market value of option = 49.74

α	γ = 0	−2	−4	−6	γ = 0	−2	−4	−6
10%	47.00	41.86	37.18	32.92	94.5%	84.2%	74.7%	66.2%
25%	43.69	33.24	24.78	18.08	87.8%	66.8%	49.8%	36.3%
50%	39.96	24.29	13.50	6.79	80.3%	48.8%	27.1%	13.6%
75%	38.04	19.32	7.87	2.49	76.5%	38.9%	15.8%	5.0%

$S = 115, T - t = 9$: Market value of option = 62.46

α	γ = 0	−2	−4	−6	γ = 0	−2	−4	−6
10%	59.17	53.00	47.34	42.16	94.7%	84.9%	75.8%	67.5%
25%	55.24	42.63	32.29	23.96	88.4%	68.2%	51.7%	38.4%
50%	50.87	31.92	18.40	9.63	81.4%	51.1%	29.5%	15.4%
75%	48.79	26.08	11.29	3.82	78.1%	41.8%	18.1%	6.1%

yield, $\hat{q}$, under the subjective process and our use of a European option pricing model. This problem is discussed and resolved in the next section on early exercise.

4.2 Subjective Incentive Effects

How much incentive does the option provide? The incentive is the change in value as perceived by the manager relative to the change in shareholder wealth; therefore, we are interested in determining the change in the *subjective* value of the option for a given change in the market value of the stock. Since our model uses the market value of the stock, the usual delta calculation on the subjective value formula provides an instantaneous measure of the incentive

$$\hat{\Delta} = e^{-\hat{q}(T-t)}\Phi(H(X)) \ . \tag{28}$$

Since $H(X)$ is smaller for the subjective process and $\hat{q} > q$, the subjective delta will be smaller than the delta given by the Black-Scholes model applied to the market price. So standard option pricing techniques will overestimate the incentive provided.

Table 2 shows the subjective delta for some typical cases. As seen in the table, the proportional effect on the deltas is smaller than on the values. Nevertheless, the subjective deltas are still substantially smaller than the objective or market deltas; therefore, standard option pricing models will often substantially overstate the incentive effects of options.

How effective are options in providing incentive from the viewpoint of the shareholders? The cost of providing an option is its market or objective value. The incentive it provides is its subjective delta. Therefore, the option's true unit price of providing incentive is the objective cost per unit of subjective delta, $C/\hat{\Delta}$.

Table 3 shows the ratio of the objective cost to the subjective delta for the options valued in Table 1 at the time of issuance. Using the Black-Scholes model we would compute a cost per unit delta of $52.57/0.842 = 62.45$. However, as seen it is much more costly than this to provide incentive to a risk-averse manager. And as the amount of stock held increases, the cost of further incentives is much higher.

The second panel shows the cost per unit of delta for a share of the company's stock with a ten-year restriction (matching the option's expiration). The objective delta of a share is one, so the cost per unit of (objective) delta is \$100, substantially more than that for the option. This is in accord with the usual sentiment that options are a cheap method of aligning manager and shareholder interests. However, as seen in the third panel this may no longer be the case when measured on a subjective basis. For very risk-averse managers who already hold substantial stock, the objective-cost-subjective-benefit trade-off favors

Table 2: **Subjective Deltas of Stock Options**

$X = 100 \quad r = 5\% \quad q = 0\% \quad \sigma = 30\% \quad T - t = 10 \quad v = 20\%$

	Subjective delta: $\hat{\Delta}$				**Subjective delta as percent of market delta:** $\hat{\Delta}/\Delta$			
	$S = 100, T - t = 10$:				Market Δ of option = 0.842			
	γ				γ			
α	0	−2	−4	−6	0	−2	−4	−6
10%	0.802	0.726	0.656	0.591	95.3%	86.3%	77.9%	70.2%
25%	0.756	0.602	0.469	0.357	89.8%	71.5%	55.7%	42.4%
50%	0.711	0.477	0.291	0.158	84.4%	56.7%	34.5%	18.7%
75%	0.699	0.416	0.193	0.067	83.0%	49.4%	22.9%	7.9%
	$S = 85, T - t = 9$:				Market Δ of option = 0.779			
	γ				γ			
α	0	−2	−4	−6	0	−2	−4	−6
10%	0.743	0.673	0.608	0.548	95.3%	86.4%	78.1%	70.4%
25%	0.700	0.556	0.433	0.329	89.8%	71.3%	55.5%	42.2%
50%	0.654	0.433	0.261	0.141	83.9%	55.6%	33.5%	18.1%
75%	0.636	0.366	0.166	0.057	81.6%	47.0%	21.3%	7.3%
	$S = 100, T - t = 9$:				Market Δ of option = 0.829			
	γ				γ			
α	0	−2	−4	−6	0	−2	−4	−6
10%	0.792	0.723	0.658	0.597	95.6%	87.2%	79.3%	72.0%
25%	0.750	0.606	0.481	0.373	90.5%	73.1%	58.0%	45.0%
50%	0.707	0.486	0.306	0.174	85.3%	58.6%	36.9%	21.0%
75%	0.694	0.425	0.208	0.078	83.7%	51.2%	25.1%	9.4%
	$S = 115, T - t = 9$:				Market Δ of option = 0.865			
	γ				γ			
α	0	−2	−4	−6	0	−2	−4	−6
10%	0.829	0.760	0.695	0.634	95.8%	87.8%	80.3%	73.3%
25%	0.788	0.645	0.519	0.410	91.0%	74.5%	60.0%	47.4%
50%	0.747	0.529	0.346	0.205	86.3%	61.2%	39.9%	23.6%
75%	0.738	0.475	0.247	0.100	85.3%	54.9%	28.6%	11.5%

Table 3: **Cost per Unit of Subjective Delta**

$S = X = 100 \quad r = 5\% \quad q = 0\% \quad \sigma = 30\% \quad T - t = 10 \quad v = 20\%$

Cost per Unit of Objective Delta
for Option: 62.45
for Restricted Stock: 100

Incentive Effectiveness of Option
Relative to Restricted Share: 160.13%

Cost per Unit of Subjective Delta
for Option

	γ			
α	0	−2	−4	−6
10%	65.55	72.37	80.13	88.99
25%	69.53	87.38	112.09	147.14
50%	73.97	110.20	180.94	333.27
75%	75.21	126.35	272.15	788.37

Cost per Unit of Subjective Delta
for Restricted Share

	γ			
α	0	−2	−4	−6
10%	103.67	111.40	119.72	128.66
25%	107.79	125.23	145.50	169.05
50%	110.52	134.99	164.87	201.38
75%	107.79	125.23	145.50	169.05

Incentive Effectiveness of Option
Relative to Restricted Share

	γ			
α	0	−2	−4	−6
10%	158.14%	153.93%	149.41%	144.57%
25%	155.02%	143.32%	129.81%	114.89%
50%	149.41%	122.49%	91.12%	60.42%
75%	143.32%	99.12%	53.46%	21.44%

using restricted shares. In extreme cases, using restricted stock can be almost five times as cost-effective as options.

Smith and Stulz [1985] and others have recognized that one problem with using stock is that its linear payoff may create an incentive for a risk-averse manager to undertake risk reducing activities even at the expense of maximizing firm value. One advantage of incentive options is they provide a convex compensation payoff that may eliminate this problem. The measure of an option's convexity is its gamma, $\Gamma \equiv \partial^2 C/\partial S^2$. Just as the subjective delta differs from the objective delta, the subjective gamma will not be properly measured by the objective gamma. The subjective gamma is

$$\hat{\Gamma} = e^{-\hat{q}(T-t)} \frac{\phi\,(H(X))}{S\sigma\sqrt{T-t}} \quad . \tag{29}$$

Unlike the delta, the Black-Scholes model of market prices can either under or overstate the convexity provided by an incentive option. For our parameter values the subjective and objective gamma are nearly identical at 0.0025. Increasing α or γ lowers the gamma and vice versa.

4.3 Subjective Risk-Taking Effects

Incentive options have also been promoted as a means to overcome a manager's hesitance to take on risky projects. A manager's personal risk aversion may make him reluctant to adopt a risky project even if it is value maximizing from the company's perspective. Since options are worth more when the underlying asset is riskier, it is argued that incentive options will counter this natural aversion.

This argument has merit even though tailoring the incentive option package to provide the precise offset may not be easy. Grant too few options and the aversion of risky projects remains. Grant too many options and the manager may have an incentive to over-invest in risky projects even at the cost of reducing firm value. In addition, it is again the incentives as measured by the subjective value which should matter, and the relation here is no longer a simple one. Taking on a new project can change both the systematic and unsystematic volatility. While only the total risk affects the market price of an option, each component has a different effect on the subjective value. Furthermore, each manager involved in the decision may have completely different subjective effects due to different risk-aversions or α.

The sensitivity of the option price to volatility is called "vega". We will measure both a total-risk vega, and an unsystematic-risk vega, for both the market and subjective values.

The total-risk vega (holding unsystematic risk constant) of the subjective value has the same functional form as the vega in the Black-Scholes model

$$\hat{\Lambda}_{\sigma|v} \equiv \left.\frac{\partial \hat{C}}{\partial \sigma}\right|_{v} = S e^{-\hat{q}(T-t)} \sqrt{T-t}\,\phi\big(H(X)\big) > 0\,. \tag{30}$$

The market-value vega is identical with $\hat{q}$ and $\hat{r}$ replaced by q and r. Both are positive; however, the market-price model will overstate (understate) the volatility sensitivity for sufficiently out-of-the-money (in-the-money) options.[10] For near-the-money options either case can be true. For example, for the parameters used in Tables 1 through 3, the true vega (per percentage point change in σ) is $\Lambda_{\sigma|v} = 0.835$. For $\alpha = 50\%$ and $\gamma = -2$, $\hat{\Lambda}_{\sigma|v} = 0.873$, and the subjective vega is larger than the objective vega for $S > \$81.87$. However, for $\alpha = 50\%$ and $\gamma = -6$, $\hat{\Lambda}_{\sigma|v} = 0.560$, and the subjective vega is larger than the objective vega only if $S > \$122.14$.

Holding total risk constant, unsystematic risk has no effect on the market price. But the subjective value is decreasing in unsystematic risk

$$\begin{aligned}
\Lambda_{v|\sigma} &\equiv \left.\frac{\partial C}{\partial v}\right|_{\sigma} = 0 \\
\hat{\Lambda}_{v|\sigma} &\equiv \left.\frac{\partial \hat{C}}{\partial v}\right|_{\sigma} = \frac{\partial \hat{C}}{\partial \hat{r}}\frac{\partial \hat{r}}{\partial v} + \frac{\partial \hat{C}}{\partial \hat{q}}\frac{\partial \hat{q}}{\partial v} \\
&= -2(T-t)(1-\gamma)\alpha v\left[\alpha X e^{-\hat{r}(T-t)}\Phi\big(h(X)\big) + (1-\alpha) S e^{-\hat{q}(T-t)}\Phi\big(H(X)\big)\right] < 0\,.
\end{aligned} \tag{31}$$

Increasing the unsystematic risk does not affect the market value of the option, but the manager is now forced to bear more risk in his stock holding since it cannot be reduced below the level α. This has a negative impact on the subjective value of the option.

It is perhaps more useful to examine the effects of systematic, β, and unsystematic risks, v, holding the other constant. Since systematic risk has no direct effect on the market or subjective values, its influence is only through changing the total risk. Increasing systematic risk while holding unsystematic risk constant will increase total risk (assuming $\beta > 0$) and, therefore, increase both the market and subjective values. In each case the vega ($\hat{\Lambda}_{\beta|v}$) is proportional to the standard vega in (30) with a proportionality factor of $\partial\sigma/\partial\beta$.

$$\Lambda_{\beta|v} = \Lambda_{\sigma|v}\frac{\beta\sigma_m^2}{\sigma} \qquad\qquad \hat{\Lambda}_{\beta|v} = \hat{\Lambda}_{\sigma|v}\frac{\beta\sigma_m^2}{\sigma} \tag{32}$$

[10] $\Lambda_{\sigma|v} \gtrless \hat{\Lambda}_{\sigma|v}$ if $S \lessgtr X \exp\left[-\left(r - q - \frac{1}{2}\sigma^2 + \alpha[\sigma^2 - \frac{1}{2}(1-\gamma)v^2]\right)(T-t)\right]$.

Holding systematic risk constant and increasing unsystematic risk will increase total risk and, therefore, increase the market value. Its effect on the subjective value is indeterminate since both total and unsystematic risk will increase.

$$\begin{aligned}
\Lambda_{\nu|\beta} &\equiv \left.\frac{\partial C}{\partial \nu}\right|_{\beta} = \left.\frac{\partial C}{\partial \sigma}\right|_{\nu} \frac{\partial \sigma}{\partial \nu} = \frac{\nu}{\sigma}\Lambda_{\sigma|\nu} > 0 \\
\hat{\Lambda}_{\nu|\beta} &\equiv \left.\frac{\partial \hat{C}}{\partial \nu}\right|_{\beta} = \left.\frac{\partial \hat{C}}{\partial \sigma}\right|_{\nu} \frac{\partial \sigma}{\partial \nu} + \left.\frac{\partial \hat{C}}{\partial \nu}\right|_{\sigma} = \frac{\nu}{\sigma}\hat{\Lambda}_{\sigma|\nu} + \hat{\Lambda}_{\nu|\sigma} \gtrless 0 \,.
\end{aligned} \tag{33}$$

Table 4 gives the various vega measures for an incentive option. As shown the market-value total-risk vega can severely misrepresent the subjective-value vega — being either too high or too low even for at-the-money options. Furthermore, unless the risk-aversion is very small, the unsystematic-risk vegas are substantially larger in absolute value than the total-risk vegas. Therefore, an increase in volatility is likely to have a negative impact on the subjective value of incentive options unless the risk is largely systematic in nature. For example, for a manager with a stock holding constraint of 50% and a risk-aversion of 5 ($\alpha = 0.5, \gamma = -4$), the total-risk and unsystematic-risk vegas are $\Lambda_{\sigma|\nu} = 0.764, \Lambda_{\nu|\sigma} = -2.244$. This means the total risk will have to increase almost three times as much as the unsystematic risk for the manager's options to even increase in subjective value. For the option's to provide actual risk-taking incentives to counter the manager's risk-aversion, the risk under consideration would have to be even more heavily weighted to systematic risk.

5 Early Exercise of Incentive Options and the Objective Value

As shown in (24), the subjective value of an incentive option is determined *as if* the dividend yield were larger and the interest rate smaller than they truly are. Because both larger dividends and lower interest rates induce call option holders to exercise their options sooner, incentive options will be optimally exercised in a fashion which appears to be suboptimally early from a market perspective. In particular, even options on stocks not paying dividends may be optimally exercised before expiration. In fact many incentive options are exercised substantially before they expire. Often they are exercised as soon as they vest.

The problem of valuing incentive options with "suboptimal" early exercise is often approximated in practice by simply using the expected time until exercise in place of the

Table 4: **Vegas of Incentive Options**

Change in value for one percentage point change in voltility

$S = X = 100 \quad r = 5\% \quad q = 0\% \quad \sigma = 30\% \quad T - t = 10 \quad v = 20\%$

Total-Risk Vega ($\Lambda_{\sigma|v}$ and $\hat{\Lambda}_{\sigma|v}$)
for Market Value: $\Lambda_{\sigma|v} = 0.764$

α \ γ	0	−2	−4	−6
10%	0.768	0.772	0.771	0.764
25%	0.783	0.797	0.775	0.721
50%	0.835	0.873	0.764	0.560
75%	0.926	1.006	0.733	0.358

Total-Risk Subjective and Objective Vega Equality Point
$\Lambda_{\sigma|v} \gtrless \hat{\Lambda}_{\sigma|v}$ for $S \lessgtr$ value below

α \ γ	0	−2	−4	−6
10%	88.69	92.31	96.08	100.00
25%	79.85	88.25	97.53	107.79
50%	67.03	81.87	100.00	122.14
75%	56.27	75.96	102.53	138.40

Unsystematic-Risk Vega ($\Lambda_{v|\sigma}$ and $\hat{\Lambda}_{v|\sigma}$)
for Market Value: $\Lambda_{v|\sigma} = 0$

α \ γ	0	−2	−4	−6
10%	−0.301	−0.819	−1.235	−1.559
25%	−0.641	−1.548	−2.032	−2.187
50%	−1.004	−2.121	−2.244	−1.766
75%	−1.201	−2.425	−2.053	−1.059

Unsystematic-Risk Vega ($\Lambda_{v|\beta}$ and $\hat{\Lambda}_{v|\beta}$)
for Market Value: $\Lambda_{v|\beta} = 0.509$

α \ γ	0	−2	−4	−6
10%	0.211	−0.304	−0.721	−1.050
25%	−0.119	−1.017	−1.515	−1.707
50%	−0.447	−1.539	−1.735	−1.392
75%	−0.584	−1.754	−1.564	−0.820

actual time to expiration in whatever (European) option pricing model is being used. The expected time until exercise is typically estimated from past experience.[11]

At best, this calculation could be used to determine the objective cost of the option to the firm or shareholders. However, even using an unbiased estimate of the expected time until exercise will not give a correct estimate of the option's value. And this method cannot be used to determine the subjective value since this will be smaller still due to the extra discounting required to compensate for the lack of diversification.

A proper calculation must recognize that the decision to exercise is endogenous. To determine the subjective value and the objective cost of the option, we must incorporate this "early" exercise using American option pricing techniques just as we would to determine the market value of an option on a stock actually paying dividends.

This "early" exercise has two effects. First since the exercise is optimal from a subjective viewpoint, it increases the subjective value of the option above that previously calculated using a European option pricing model. Second since the exercise is premature relative to a market valuation, it reduces the objective cost of the option to below the market value of a comparable freely traded option.

No formula is known for valuing American options with early exercise; however, a number of approximations are available. We will use the "barrier-derivative" approximation developed in Ingersoll [1998]. This method has been shown to be extremely accurate relative to other commonly employed methods — particularly for long-term options. In addition it can be readily modified to include vesting which restricts the exercise of the option for some time after its initial grant. It can be used to determine all three values, the subjective value, the market value, and the objective cost. In general all three of these values will differ when early exercise is considered.

The value of an American option depends on the policy used to exercise it. The exercise policy is characterized by a stock price at each point of time such that the call option is exercised if the stock price is at or above that value. Let $K(t)$ denote the exercise policy, then the value of the option is the present value of $\text{Max}[S_T - X, 0]$ if the stock price never rises to $K(t)$ plus the present value of $K(t) - X$ received the first time the stock prices reaches $K(t)$. The policy that maximizes this present value is the optimal exercise policy, and the present value under that policy is the value of the American option. This valuation problem is a barrier option problem with an unknown barrier.

The barrier-derivative approximation method posits a parametric class of exercise policies represented by a barrier and computes the present value of exercising the option when

[11] This calculation generally reduces the computed value of the option; though it need not do so if a European option pricing model has been employed and dividends are being paid.

the stock price hits this barrier or when it expires in-the-money if it does not hit the barrier. It then chooses the parameters that maximize the present value. This method determines an approximation which is a lower bound to the actual value.[12]

We solve this problem using the class of constant exercise policies. We choose constant policies for three reasons. (i) The valuation can be accomplished mostly with analytic techniques, and, as shown in Ingersoll [1998], the best "constant" exercise policy gives a value very close to the true value. (Errors are approximately 0.1% to 0.2% for ten-year options.) (ii) We are not actually looking for the optimal policy, but a policy like one that a typical manager actually adopts. (iii) This class is easily modified to allow for vesting.

The approximate value of the option computed for the constant exercise policy is

$$C \gtrapprox C^*_{\text{barr}} = \operatorname*{Max}_{k} C_{\text{barr}}(S,t;T;k)$$

where

$$\begin{aligned} C_{\text{barr}} \equiv \mathcal{S}(S,t;T;\{S_T > X\}\,\&\{S_{\max} < k\}) - X\mathcal{D}(S,t;T;\{S_T > X\}\,\&\{S_{\max} < k\}) \\ + (k - X)\mathcal{T}(S,t;T;k)\ . \end{aligned} \tag{34}$$

$\mathcal{S}(S,t;T;\mathcal{E})$ and $\mathcal{D}(S,t;T;\mathcal{E})$ are a digital share and a digital option, respectively. They are the present values at time t of receiving S_T and \$1 at time T if the event $\mathcal{E}$ occurs. $\mathcal{T}(S,t;T;k)$ is a first-touch digital. It is the present value at time t of receiving \$1 the first time (before expiration at T) that the stock price reaches the barrier k. If the stock price never reaches the barrier and is in-the-money at expiration, then the option is exercised for $S_T - X$. The present value of this exercise is given by the first two terms. If the first touch at the barrier occurs before the option expires, then it is immediately exercised for $k - X$. The present value of this is given by the last term.

As shown in Ingersoll [2000], the formulas for these three digital contracts are

$$\begin{aligned} &\mathcal{S}(S,t;T;\{S_T > X\}\,\&\{S_{\max} < k\}) \\ &\quad = Se^{-q(T-t)}\left\{\Phi(H(X)) - \Phi(H(k)) - (k/S)^{2(\lambda+1)}\left[\Phi(H(XS^2/k^2)) - \Phi(H(S^2/k))\right]\right\} \end{aligned}$$

[12] If the class includes the optimal exercise policy, then the value given will be correct and not an approximation. If it does not, then the exercise policy will be suboptimal and the resulting value will be lower than the true value. Even if the policy is apparently far from optimal in appearance, the resulting value can be very close to the correct value as shown in Ingersoll [1998].

$$\mathcal{D}(S,t;T;\{S_T > X\}\,\&\{S_{\max} < k\})$$
$$= e^{-r(T-t)}\left(\Phi(h(X)) - \Phi(h(k)) - (k/S)^{2\lambda}\left[\Phi(h(XS^2/k^2)) - \Phi(h(S^2/k))\right]\right)$$
$$\mathcal{T}(S,t;T;k) = (k/S)^{\lambda-\kappa}\Phi(H_\kappa(k)) + (k/S)^{\lambda+\kappa}\Phi(h_\kappa(k)) \tag{35}$$
where $$H_\kappa(z) \equiv \frac{\ell\text{n}(S/z) + \kappa\sigma^2(T-t)}{\sigma\sqrt{T-t}} \qquad h_\kappa(z) \equiv H_\kappa(z) - 2\kappa\sigma\sqrt{T-t}$$
$$\lambda \equiv \frac{\hat{r}-\hat{q}}{\sigma^2} - \frac{1}{2} \qquad \kappa \equiv \sqrt{\lambda^2 + 2\hat{r}/\sigma^2}\,.$$
and $H(z)$ and $h(z)$ are defined in (27).

The market value, the subjective value, and the objective value can all be computed with this method. Using the actual risk-neutral stochastic process in (34) and (35) will give $C_{\text{barr}}(S,t;k^*)$, the value of the option if it were freely marketable. Using the risk-neutral process, $dS/S = (\hat{r}-\hat{q})dt + \sigma\, d\omega$, will give the subjective value of the option, $\hat{C}_{\text{barr}}(S,t;\hat{k}^*)$. The value-maximizing exercise choices for the actual and subjective processes will also differ, $k^* \neq \hat{k}^*$.[13] Finally, the objective cost of the option is determined using the subjective value maximizing policy with the actual risk-neutral process, $C_{\text{barr}}(S,t;\hat{k}^*)$. This is the cost that the company perceives in the option under the exercise policy actually adopted by the manager.

Table 5 shows the objective and subjective values of incentive options under various conditions at issuance and one year later. The "value of marketed option" and the "European objective value of option" at the top of each panel are values equivalent to the market values in the Table 1. The exact numbers differ because the dividend yield is now 1%. The European objective value is a bit below the marketed value since it does not allow optimal early exercise. The "European subjective" values in the top row of each panel are equivalent to the subjective values in the Table 1. As before, the subjective value is less than the objective value. The differences increase with an increase in the stock-holding restriction, α, or risk aversion, $1-\gamma$.

The true subjective value to the manager is given in the second row of each panel. This is the value to the manager if he follows the optimal early exercise policy which is approximated by the constant policy ($\hat{k}^*$ on the last line). This value is always larger than the European subjective value since the manager can always choose not to exercise. For cases when the European subjective value is very low (tight restriction or high risk aversion), the option is usually exercised quite a bit earlier than would an equivalent marketed option, and the true subjective value is substantially larger than the European subjective value. For example, for $\gamma = -6$, $\alpha = 75\%$, and $S = 115$, the true subjective value is more than six

[13] As with the true optimal policies, the option is exercised sooner under the subjective process, $k^* < \hat{k}^*$.

Table 5: **Objective and Subjective Values of Stock Options with Early Exercise**

$$X = 100 \quad r = 5\% \quad q = 1\% \quad \sigma = 30\% \quad v = 20\%$$

$S = 100, T - t = 10$: Value of marketed option = 44.83 ($k^* = 666$)
European objective value of option = 44.68

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	28.67	20.73	14.62	20.29	10.57	4.91	15.75	5.78	1.59
True Subj.	31.52	25.84	21.59	25.11	18.22	13.74	21.33	13.98	9.81
Eu. ($T = \mathbb{E}[\tilde{t}_k]$)	41.72	39.62	37.62	39.33	35.65	32.29	37.49	32.48	28.22
True Obj.	42.05	38.94	35.74	38.48	32.56	27.29	35.53	27.58	21.39
$\mathbb{E}[\tilde{t}_k]$	8.53	7.61	6.81	7.49	6.09	4.99	6.76	5.05	3.85
$\hat{k}^*$ (True Subj.)	255	207	181	202	164	145	180	146	131

$S = 85, T - t = 9$: Value of marketed option = 32.12 ($k^* = 647$)
European objective value of option = 32.07

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	20.55	14.85	10.49	14.33	7.43	3.46	10.82	3.93	1.09
True Subj.	22.01	17.58	14.22	16.91	11.48	7.93	13.83	8.08	4.83
Eu. ($T = \mathbb{E}[\tilde{t}_k]$)	30.31	29.04	27.86	28.84	26.72	24.88	27.72	24.95	22.73
True Obj.	30.40	28.38	26.29	28.04	24.14	20.59	26.03	20.73	16.46
$\mathbb{E}[\tilde{t}_k]$	8.14	7.56	7.06	7.48	6.59	5.88	7.00	5.91	5.12
$\hat{k}^*$ (True Subj.)	245	200	177	195	161	144	175	144	130

$S = 100, T - t = 9$: Value of marketed option = 42.82 ($k^* = 654$)
European objective value of option = 42.72

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	28.13	20.75	14.97	20.23	10.98	5.37	15.81	6.18	1.87
True Subj.	30.50	25.18	21.15	24.44	17.91	13.61	20.81	13.82	9.77
Eu. ($T = \mathbb{E}[\tilde{t}_k]$)	39.96	37.97	36.09	37.68	34.22	31.07	35.91	31.23	27.23
True Obj.	40.33	37.50	34.58	37.06	31.62	26.71	34.31	26.95	21.11
$\mathbb{E}[\tilde{t}_k]$	7.75	6.95	6.25	6.84	5.61	4.63	6.19	4.68	3.59
$\hat{k}^*$ (True Subj.)	249	203	179	199	163	145	177	145	130

$S = 115, T - t = 9$: Value of marketed option = 54.17 ($k^* = 660$)
European objective value of option = 54.01

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	36.37	27.27	20.02	26.82	15.10	7.70	21.54	8.97	2.90
True Subj.	39.89	33.89	29.43	33.12	25.96	21.46	29.17	21.70	17.79
Eu. ($T = \mathbb{E}[\tilde{t}_k]$)	50.08	47.23	44.43	46.83	41.59	36.51	44.24	36.82	29.85
True Obj.	50.82	47.08	43.23	46.53	39.40	32.99	42.98	33.36	25.80
$\mathbb{E}[\tilde{t}_k]$	7.31	6.26	5.34	6.12	4.52	3.24	5.29	3.31	1.91
$\hat{k}^*$ (True Subj.)	253	206	181	202	164	145	180	146	131

times as large as the European subjective value — a difference which is more than 13% of the stock price.

The third row of each panel shows a value commonly calculated in practice. A European pricing model is used with a time to maturity set equal to the expected time until the option is exercised. These times are usually estimated based on past experience. Here we compute the risk-neutral expected time until the optimal (constant) exercise barrier is reached. Since the barriers depend on the restriction and the risk aversion, the expected time to exercise also differs with these parameters. The optimal barriers and the expected time until exercise are given in the sixth and fifth row of each panel.[14]

The fourth value in each panel is the true objective cost. This is the present value to the company shareholders of the future payout. As they do not perceive the "discount" in value due to a lack of diversification, it is computed using the optimal subjective exercise policy with the *objective* risk-neutral evolution. This value is less than the marketed value of the option because it recognizes that the manager follows an exercise policy which is suboptimal in objective terms. Nevertheless, it is still substantially in excess of the subjective value. The table shows that the commonly used practice of estimating the value with an adjusted expiration date does reasonably well only when the stock-holding restriction and relative risk aversion are small.

This table may also help explain an empirical anomaly cited in the literature, namely that CEOs seem to exercise their options earlier than do other executives. André, Boyer, and Gagné [2001] report that "non-CEO executives seem to exercise their stock options about a calendar year than the CEO" and that they are "more likely to exercise when a new CEO has been appointed." The explanation they give a based on a tournament model. However, the model here may also supply an answer. If CEOs hold a greater fraction of their wealth in the company stock (have a higher α), then we should expect them to exercise earlier. For example, if a CEO has 75% of his wealth in a company's stock and another executive has

[14] The expected time until the barrier is hit can be easily determined from the first-touch digital's value. Let $\psi(t_k)$ be the probability density of the fist-passage time to k, then the first-touch digital's value is the expected discounted value of \$1 where the expectation is over the time until payment

$$\mathcal{T}(S,t;k) = \int_t^T e^{-r(t_k-t)}\psi(t_k)\,dt_k\,.$$

With a suitable reinterpretation of the parameters, this will be recognized as the moment generating function of the distribution. The expected value can be computed in the usual fashion

$$\begin{aligned}\mathbb{E}[\tilde{t}_k] =&(T-t)\left[\Phi(-h(k)) - (k/S)^{2\lambda}\Phi(-h(S^2/k))\right] + \frac{\ell\text{n}(k/S)}{\lambda\sigma^2}\left[\Phi(h(k)) - (k/S)^{2\lambda}\Phi(-h(S^2/k))\right]\\ &- \frac{\sqrt{T-t}}{\lambda\sigma}\left[\phi(h(k)) - (k/S)^{2\lambda}\phi(-h(S^2/k))\right]\,.\end{aligned}$$

Table 6: **Objective and Subjective Deltas of Stock Options with Early Exercise**

$X = 100 \quad r = 5\% \quad q = 1\% \quad \sigma = 30\% \quad v = 20\%$

$S = 100, T - t = 10:$ Delta of marketed option = 0.74
European objective delta of option = 0.74

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	0.52	0.40	0.30	0.40	0.24	0.13	0.35	0.15	0.05
True Subj.	0.60	0.54	0.51	0.54	0.48	0.45	0.51	0.45	0.42
Eu. ($T = \mathbb{E}[\tilde{t}_K]$)	0.73	0.73	0.72	0.72	0.71	0.70	0.72	0.70	0.68
True Obj.	0.68	0.62	0.56	0.62	0.51	0.41	0.56	0.42	0.30

$S = 85, T - t = 9:$ Delta of marketed option = 0.69
European objective delta of option = 0.68

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	0.48	0.37	0.28	0.37	0.22	0.11	0.31	0.13	0.04
True Subj.	0.53	0.47	0.42	0.46	0.38	0.31	0.42	0.32	0.25
Eu. ($T = \mathbb{E}[\tilde{t}_K]$)	0.67	0.67	0.66	0.67	0.65	0.64	0.66	0.64	0.62
True Obj.	0.64	0.59	0.54	0.58	0.49	0.40	0.54	0.41	0.31

$S = 100, T - t = 9:$ Delta of marketed option = 0.74
European objective delta of option = 0.73

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	0.53	0.42	0.32	0.42	0.26	0.14	0.36	0.17	0.06
True Subj.	0.60	0.54	0.51	0.54	0.48	0.45	0.51	0.45	0.42
Eu. ($T = \mathbb{E}[\tilde{t}_K]$)	0.73	0.72	0.71	0.72	0.71	0.70	0.71	0.70	0.68
True Obj.	0.68	0.63	0.57	0.62	0.51	0.41	0.57	0.42	0.31

$S = 115, T - t = 9:$ Delta of marketed option = 0.78
European objective delta of option = 0.77

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	0.57	0.45	0.35	0.46	0.29	0.17	0.40	0.20	0.08
True Subj.	0.65	0.62	0.60	0.62	0.59	0.60	0.60	0.60	0.66
Eu. ($T = \mathbb{E}[\tilde{t}_K]$)	0.77	0.77	0.76	0.76	0.76	0.75	0.76	0.75	0.75
True Obj.	0.71	0.65	0.59	0.64	0.53	0.42	0.59	0.43	0.31

Table 7: **Cost per Unit of Subjective Delta (with Early Exercise)**

$S = X = 100 \quad T - t = 10 \quad r = 5\% \quad q = 1\% \quad \sigma = 30\% \quad \upsilon = 20\%$

Cost per Unit of Market Delta: 60.38

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	-2	-4	-6	-2	-4	-6	-2	-4	-6
Eu. model	85.99	111.54	148.26	110.33	186.13	353.60	129.05	291.13	888.91
True model	70.25	71.50	70.50	71.02	67.61	61.09	69.33	61.24	50.82
$T = \mathbb{E}[\tilde{t}_K]$	57.06	54.61	52.27	54.27	49.98	46.05	52.12	46.28	41.24

only 50%, then we would expect to see the CEO exercise 1.4, 1.0 or 1.1 years earlier than the other executive if they both had the same risk aversion ($\gamma = -2, -4$, and -6, respectively). Furthermore, we would not expect them to wait longer than a newly appointed CEO who would likely have a much smaller stock holding.

The deltas of the option calculated with the same models are given in Table 6. As with the values, we see that ignoring the early exercise of incentive options can vastly understate their deltas. For realistic parameter values, the true subjective deltas can be many times larger than those computed with a European model. Using the expiration-adjusted European option pricing model is reasonably accurate only for small stock holding restrictions and risk aversions. In other cases it vastly overstates the subjective deltas as do the European and market models.

Table 7 shows the cost per unit delta for the option when first issued calculated with various models. The parameters are the same as used in Tables 5 and 6. The "market" cost per unit "market" delta is \$60.38. Using a European option pricing model ignoring early exercise gives an objective cost per unit of subjective delta that ranges from \$85.99 to \$888.91 for the parameters considered. The apparent cost is higher for a larger stock holding restriction or greater risk aversion because the subjective delta is lower in those cases.

The true objective cost per unit of subjective delta is substantially lower — ranging from \$50.82 to \$70.25, because the true subjective delta of this option is substantially above the European-model subjective value. And an increase in the stock holding restriction or risk aversion can either increase or decrease the cost per unit of subjective delta. Furthermore because the objective cost is affected as well as the subjective delta, the market-based model can either over or under estimate the true effectiveness of options in providing incentives.

6 Restrictions on Exercising: Vesting and Termination

Two features of incentive options that affect early exercise are vesting and termination of employment. An incentive option cannot be exercised until it vests and can only be exercised while the manager is employed or during any post-employment grace period for exercise. The grace period is commonly one to three months, but longer periods are also seen. Some incentive options continue to be exercisable up to their original expiration date. This is most commonly true in the case of retirement. Generally, if an employee is terminated before his options vest, he loses al rights to them although there have been several court cases which have found the opposite. In this section we examine the effects of vesting and termination. Since these issues have been considered before, we confine our attention to how they interact with the subjective valuation models developed here.

6.1 Vesting

There are a number of different schemes used to vest options. The most common are cliff vesting, straight vesting, stepped vesting, and performance vesting. With cliff vesting, all options granted on a given date vest after a set period of time, usually two to four years. With straight and stepped vesting, options granted in a given year vest gradually over time. For straight vesting the same proportion vests each year. For stepped vesting a different proportion vests each year. For example, 25% of the options would vest after each year for four year with straight vesting and 10%, 20%, 30% and 40% might vest each year with stepped vesting. Performance vesting links the vesting of the options to meeting certain targets in sales, income, etc. Many options also vest sooner or even immediately in case of a sale, IPO, merger, or other similar event for the firm.

Vesting clearly reduces the subjective value of the option since it restricts when exercise can occur. For the same reason, it reduces the “market” value of the option — the value it would have if marketable. The market value is affected less than the subjective value since incentive options are often exercised much earlier than marketable options would be an are therefore more apt to run afoul of the vesting rule. The objective cost of the option can either increase or decrease. In particular, if the manager wishes to exercise his option very early under the optimal subjective exercise policy, vesting may force him to delay doing so and increase the objective cost of the option.

To determine the value of an option with cliff vesting, we solve the barrier problem using the class of policies which preclude exercise prior to vesting at time T°, and are constant after the option vests. That is, for a constant policy approximation, exercise occurs when the stock price first reaches $K(t)$ for[15]

$$K(t) = \begin{cases} \infty & t \le T^\circ\ , \\ k & t > T^\circ\ . \end{cases} \tag{36}$$

The approximate value of the option computed for the constant exercise policy in (36) is

$$C \gtrapprox C^*_{\text{barr-v}} = \operatorname*{Max}_k C_{\text{barr-v}}(S,t;T;k)$$

where

$$\begin{aligned} C_{\text{barr-v}} \equiv\, & \mathcal{S}(S,t;T^\circ;\{S_{T^\circ} > k\}) \ - X\mathcal{D}(S,t;T^\circ;\{S_{T^\circ} > k\}) \\ & + \mathcal{S}(S,t;T;\{S_T > X\}\,\&\{S_{\max(T^\circ,T)} < k\}) \\ & - X\mathcal{D}(S,t;T;\{S_T > X\}\,\&\{S_{\max(T^\circ,T)} < k\}) \\ & + (k-X)\mathcal{T}(S,t;T;k\,\&\,S_{T^\circ} < k)\ . \end{aligned} \tag{37}$$

If the stock price is above k when the option vests at time T°, then it is exercised immediately for $S_{T^\circ} - X$. The present value of this exercise is given in the digitals in the first two terms. These digitals mature at time T°. If the stock price never reaches the barrier and is in-the-money at expiration, then the option is exercised for $S_T - X$. The present value of this exercise is given by the digitals in the second two terms. These digitals mature at time T°. If the first touch at the barrier occurs before the option expires, then it is immediately exercised for $k - X$. The present value of this is given by the last term.

[15] $K(T^\circ) = \infty$ rather than k since exercise at $t = T^\circ$ is handled directly with the first two terms in (37) rather than with the barrier to properly account for the exercised value when $S_{T^\circ} > k$.

The formulas for the digitals used here are

$$
\begin{aligned}
&\mathcal{S}(\cdot;\{S_T > X\}\,\&\,\{S_{\max(T^\circ,T)} < k\}) \\
&\quad = Se^{-q(T-t)}\Big\{\Phi_2(-H(k), -H^\circ(k), \rho) - \Phi_2(-H(X), -H^\circ(k), \rho) \\
&\qquad - (k/S)^{2(\lambda+1)}\Big[\Phi_2(-H(S^2/k), H^\circ(S^2/k), -\rho) - \Phi_2(-H(XS^2/k^2), H^\circ(S^2/k), -\rho)\Big]\Big\} \\
&\mathcal{D}(\cdot;\{S_T > X\}\,\&\,\{S_{\max(T^\circ,T)} < k\}) \\
&\quad = e^{-r(T-t)}\Big\{\Phi_2(-h(k), -h^\circ(k), \rho) - \Phi_2(-h(X), -h^\circ(k), \rho) \\
&\qquad - (k/S)^{2\lambda}\Big[\Phi_2(-h(S^2/k), h^\circ(S^2/k), -\rho) - \Phi_2(-h(XS^2/k^2), h^\circ(S^2/k), -\rho)\Big]\Big\} \\
&\mathcal{T}(\cdot;k\,\&\,S_{T^\circ} < k) = (k/S)^{\lambda-\kappa}\Phi_2(-H_\kappa(k), H_\kappa^\circ(k), -\rho) + (k/S)^{\lambda+\kappa}\Phi_2(-h_\kappa(k), h_\kappa^\circ(k), -\rho) \\
&\qquad \text{where} \qquad \rho \equiv \sqrt{(T^\circ - t)/(T-t)}\,,
\end{aligned}
\tag{38}
$$

$\Phi_2(\cdot)$ is the standard bivariate cumulative normal function, and the other quantities are defined in (27) and (35). H and h functions with a $\circ$ superscript are evaluated with an expiration date of T°.

Table 8 compares the values of options which vest immediately and after one, two, three, and four years. The values of straight and stepped vesting options can be computed as equal or appropriately weighted averages of the values of the cliff-vested options given here.

Obviously, vesting has no effect on the European objective or subjective values of the option. It has very little effect (less than one cent) on the market value since, for these parameter values, it is very unlikely it would be optimal to exercise the option before it vests. If the dividend yield were higher, the volatility lower, or the maximum vesting period longer, then vesting would affect the market value more.

The big effect of vesting is on the true subjective and objective values. Vesting must decrease the subjective value of the option, and, as seen in the table, the magnitude can be significant. For a manager with a stock holding restriction of 50% and a relative risk aversion of 5, vesting after four years reduces the subjective value of the option by more than 10%. For a manager with $\alpha = 75\%$ and $\gamma = -6$, the reduction in value is more than 43%. Straight or stepped vesting would have a proportionately smaller effect. On the other hand, for the parameters considered here, vesting always increases the objective cost of the option. For example for $\alpha = 50\%$ and $\gamma = -4$, and $\alpha = 75\%$ and $\gamma = -6$, four-year vesting increases the objective cost by 16% and 59%, respectively.

Table 8: **Objective and Subjective Values of Stock Options with Vesting**

$S = X = 100 \quad T - t = 10 \quad r = 5\% \quad q = 1\% \quad \sigma = 30\% \quad \upsilon = 20\%$

$T^\circ - t = 0$:

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
True Subj.	31.52	25.84	21.59	25.11	18.22	13.74	21.33	13.98	9.81
True Obj.	42.05	38.94	35.74	38.48	32.56	27.29	35.53	27.58	21.39

$T^\circ - t = 1$:

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
True Subj.	31.52	25.84	21.57	25.11	18.16	13.48	21.32	13.77	9.12
True Obj.	42.05	38.96	35.88	38.51	32.96	28.57	35.65	28.77	24.31

$T^\circ - t = 2$:

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
True Subj.	31.51	25.78	21.36	25.05	17.78	12.65	21.17	13.06	7.84
True Obj.	42.09	39.27	36.69	38.86	34.38	31.13	36.45	31.25	28.06

$T^\circ - t = 3$:

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
True Subj.	31.47	25.57	20.91	24.85	17.15	11.62	20.85	12.16	6.62
True Obj.	42.26	39.88	37.84	39.53	36.06	33.62	37.61	33.69	31.30

$T^\circ - t = 4$:

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
True Subj.	31.34	25.21	20.29	24.51	16.37	10.55	20.39	11.21	5.54
True Obj.	42.54	40.63	39.07	40.34	37.70	35.88	38.84	35.91	34.12

For all vesting periods:

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	28.67	20.73	14.62	20.29	10.57	4.91	15.75	5.78	1.59
Eu. ($T = \mathbb{E}[\tilde{t}_k]$)	41.72	39.62	37.62	39.33	35.65	32.29	37.49	32.48	28.22†
Eu. Obj.	44.68 for all parameter values								
Mkt. Value	44.83 for all parameter values								

† For $\gamma = -6$ and $\alpha = 75\%$, the expected time until the optimal subjective barrier is hit is 3.85 years. For the option which vests after four years, a maturity of four should be used instead of 3.85. This gives a value of 28.80 in place of 28.22.

Table 9 compares the subjective deltas of options which vest immediately and after one, two, three, and four years. It also gives the objective cost per unit of subjective delta for these options.

The effects on the deltas are similar to those on the subjective value. In particular, the subjective delta is a decreasing function of the duration of the vesting period. So since the objective cost is generally increased by vesting and increased more with a longer vesting period, vesting significantly increases the cost per unit of subjective delta.

As shown in the table, the delta can be as much as four times smaller than the market delta for reasonable parameters and the cost per unit delta can be as great as three times as large as the cost measured using market data.

6.2 Termination of Employment

To determine the effects of early exercise due to termination of employment, we must alter the original pricing equation. Let $\pi(t)$ denote the risk-neutral probability that the option must be exercised at time t due to termination, and let $\pi(T) = 1 - \int_0^{T^-} \pi(s)\,ds$ denote the probability that the option expires before employment termination. When termination occurs the option must be exercised for $S - X$ or discarded if it is out of the money. If the option has not yet vested, then (with a few exceptions) it cannot be exercised.

The pricing equation as altered to account for termination is[16]

$$
\begin{aligned}
& 0 = \tfrac{1}{2}\sigma^2 S^2 F_{SS} + (\hat{r} - \hat{q})F_S - \hat{r}F + F_t + \pi(t)\,[\Psi(S,t) - F] \\
\text{where} \qquad & \Psi(S,t) = \begin{cases} \text{Max}(S - X, 0) & \text{if } t \geq T^\circ, \\ 0 & \text{if } t < T^\circ. \end{cases}
\end{aligned}
\tag{39}
$$

This equation is identical to (24) except for the final term. This term captures the expected change in the value of the option from its "alive" value, F, to its "terminated" value, $\Psi(\cdot)$. This change occurs at time t with probability $\pi(t)$.

If the probability of termination is independent of the stock price process, then the subjective and objective values of the option can be determined as a probability-weighted average of the values of options with different maturities.

$$
C_{\text{w/term.}} \gtrapprox \underset{k}{\text{Max}} \left[\int_{T^\circ}^{T} \pi(\tau) C_{\text{barr-v}}(S, t; \tau; k)\, d\tau \right] \tag{40}
$$

[16] If there is an exercising grace period of duration τ after termination during which the option must be exercised, then $\pi(t)$ is equal to the probability of termination occurring at time $t - \tau$, and the conditions for $\Psi(S,t)$ are $t \gtrless T^\circ + \tau$. Of course, the option can also be exercised voluntarily during the grace interval.

Table 9: **Subjective Deltas of Stock Options with Vesting**

$S = X = 100 \quad T - t = 10 \quad r = 5\% \quad q = 1\% \quad \sigma = 30\% \quad v = 20\%$

Subjective Delta

Delta of Marketable option = 0.74†

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
$T^\circ - t = 0$	0.60	0.54	0.51	0.54	0.48	0.45	0.51	0.45	0.42
$T^\circ - t = 1$	0.60	0.54	0.50	0.54	0.48	0.42	0.51	0.43	0.36
$T^\circ - t = 2$	0.60	0.54	0.49	0.54	0.45	0.37	0.50	0.39	0.28
$T^\circ - t = 3$	0.59	0.53	0.47	0.53	0.42	0.33	0.48	0.35	0.23
$T^\circ - t = 4$	0.59	0.51	0.45	0.51	0.39	0.29	0.47	0.31	0.18

Objective Cost per Unit of Subjective Delta

Market Cost per Unit of Delta on Marketable option = 60.38†

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
$T^\circ - t = 0$	70.25	71.50	70.50	71.02	67.61	61.09	69.33	61.24	50.82
$T^\circ - t = 1$	70.25	71.57	71.07	71.10	69.34	67.69	69.77	66.97	68.30
$T^\circ - t = 2$	70.38	72.80	74.75	72.41	76.11	83.20	72.81	80.55	99.06
$T^\circ - t = 3$	71.03	75.46	80.56	75.12	85.21	102.15	77.57	96.80	137.57
$T^\circ - t = 4$	72.20	79.06	87.60	78.74	95.67	124.08	83.25	115.11	185.21

† The market value and delta of the option are affected by vesting; however, the differences in value are less than 0.01.

where $C_{\text{barr-v}}(\cdot)$ is defined in (37). Note that a single exercise policy must be chosen, not one for each expiration date; i.e., the integral is maximized and not each integrand separately. Of course, the integral may be interpreted as a sum when appropriate.

The qualitative effect of termination can be seen in the solution provided above. The possibility of termination is similar to reducing the time to expiration of an option. This, of course, reduces the various values and means that early exercise is even more likely since the "penalty" for exercising (surrendering the remaining option value) is now less severe.

Table 10 shows the subjective and objective values of incentive options for a manager with a 5% probability of termination each year. This possibility of termination reduces the market value of a ten-year option by 19% from \$44.83 to \$36.30 because the option might have to be exercised before it is optimal to do so — or even discarded before expiration.

The percentage effects on the objective and subjective value of the option are a bit less because the incentive option will be exercised earlier than would an equivalent marketed option so termination-forced exercise is less likely to occur. For high stock holding restrictions or high risk aversion, the termination effect is smallest. Again the reason is that under these conditions the optimal exercise occurs earlier.

For options that vest after four years, the termination possibility has a larger effect on all values. If termination occurs before the option has vested, then it cannot be exercised even if it is in the money. This loss reduces the value of the marketed option to 74.5% of the no-termination value. The subjective and objective values are also reduced more than for an option that vests immediately but less than is the market value.

7 Indexed Incentive Options

In the next two sections we illustrate how the methods developed in this paper can be used to analyze incentive option problems. The examples we choose are indexed options and option repricing. Indexing incentive options has many advocates who argue it reduces the cost of incentive options and does not reward managers for market-based gains in value or, conversely, penalize them for market-based losses in value. Option repricing has become prevalent in recent years and has drawn much criticism.

An indexed option is one whose strike price rises or falls along with some portfolio or index. The usual choices for the portfolio are the market or some industry or sector portfolio. Let the evolution of the index be

$$\begin{aligned} dI/I &= (\mu_I - q_I)dt + \sigma_I \, d\omega_I \\ &= (\mu_I - q_I)dt + \beta_I \sigma_m \, d\omega_m + \psi \nu \, d\omega + \upsilon \, d\omega_\epsilon \, . \end{aligned} \tag{41}$$

Table 10: **Objective and Subjective Values of Stock Options with Employee Termination**

$S = X = 100 \quad T - t = 10 \quad r = 5\% \quad q = 1\% \quad \sigma = 30\% \quad \nu = 20\%$

Termination Probability 5% each year

Immediate Vesting ($T^{\circ} - t = 0$)

Value of Marketable Option = 36.30
(81.0% of No-Termination Value)

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
True Subj.	26.30	21.92	18.59	21.27	15.85	12.26	18.21	12.40	8.98
True Obj.	34.37	32.08	29.68	31.72	27.25	23.23	29.45	23.41	18.65

Percent of No-Termination Value

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
True Subj.	83.4%	84.8%	86.1%	84.7%	87.0%	89.3%	85.4%	88.7%	91.5%
True Obj.	81.7%	82.4%	83.1%	82.4%	83.7%	85.1%	82.9%	84.9%	87.2%

Vesting after Four Years ($T^{\circ} - t = 4$)

Value of Marketable Option =33.40
(74.5% of No-Termination Value)

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
True Subj.	23.90	19.37	15.66	18.81	12.65	8.19	15.65	8.67	4.32
True Obj.	32.06	30.96	30.02	30.78	29.15	27.97	29.86	27.98	26.79

Percent of No-Termination Value

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
True Subj.	76.3%	76.8%	77.2%	76.7%	77.3%	77.7%	76.8%	77.3%	77.9%
True Obj.	75.4%	76.2%	76.8%	76.3%	77.3%	78.0%	76.9%	77.9%	78.5%

If the strike price is linked to the total return on the index, then $q_I = 0$ regardless of any dividends actually paid on the index.

The second line decomposes the risk of the index into its market risk, the risk correlated with the nonsystematic risk of the stock, and the risk uncorrelated with both. The three Wiener processes in the second line are independent, and ψ is the regression coefficient of the index on the nonsystematic risk of the stock; i.e., $\psi \equiv \mathrm{Cov}[dI/I,\ dS/S - \beta dM/M]/v^2$. If the index is the market, then $\psi = \upsilon = 0$ and the index has no nonsystematic risk correlated or uncorrelated with the stock.

The two-factor subjective pricing equation equivalent to (24) for derivatives on the stock and the index can be derived as before. By Itô's lemma

$$
\begin{aligned}
0 =& \mathbb{E}[d(\Theta F(S,I,t))] \\
=& \mathbb{E}\Big[\Theta F_S\, dS + \Theta F_I\, dI + \Theta F_t\, dt + \tfrac{1}{2}\Theta F_{SS} dS^2 + \Theta F_{SI}\, dS\, dI + \tfrac{1}{2}\Theta F_{II} dI^2 \\
&\quad + F\, d\Theta + d\Theta\, F_S\, dS + d\Theta\, F_I\, dI\Big] \\
=& \Theta\Big[(\mu - q)SF_S + (\mu_I - q_I)IF_I + F_t + \tfrac{1}{2}\sigma^2 S^2 F_{SS} + \rho\sigma\sigma_I SIF_{SI} + \tfrac{1}{2}\sigma_I^2 I^2 F_{II} \\
&\quad - [r - (1-\gamma)\alpha^2 v^2]F - [\beta(\mu_m - r) + (1-\gamma)\alpha v^2]SF_S \\
&\quad - [\beta_I(\mu_m - r) + (1-\gamma)\alpha\psi v^2]IF_I\Big]\, dt\ .
\end{aligned}
\tag{42}
$$

So the two-factor subjective partial differential pricing equation is

$$
\begin{aligned}
0 = \frac{1}{2}\sigma^2 S^2 F_{SS} + \rho\sigma\sigma_I SIF_{SI} + \frac{1}{2}\sigma_I^2 I^2 F_{II} + (\hat{r} - \hat{q})SF_S + (\hat{r} - \hat{q}_I)IF_I - \hat{r}F + F_t \\
\text{where} \qquad \hat{q}_I \equiv q_I + (1-\gamma)\alpha(\psi - \alpha)v^2
\end{aligned}
\tag{43}
$$

and the other variables are as defined in (24).[17]

The most common type of indexed option is the out-performance option with a payoff when exercised at time t of $\mathrm{Max}[S_t - XI_t/I_0, 0]$. Using Margrabe's [1978] option-to-exchange formula, the European subjective value of an out-performance option is

$$
\begin{aligned}
\hat{C}(S,I,t) = \mathcal{C}(Se^{-\hat{q}(T-t)}, t; T, IX/I_0, \hat{q}_I, \Upsilon) \\
\text{where} \qquad \Upsilon^2 \equiv \sigma^2 - 2\rho\sigma\sigma_I + \sigma_I^2 = (\beta - \beta_I)^2\sigma_m^2 + v^2(1-\psi)^2 + \upsilon^2\ .
\end{aligned}
\tag{44}
$$

[17] The correlation coefficient, ρ, is the total correlation between the stock and index, $\rho \equiv \mathrm{Cov}[dS/S, dI/I]/(\sigma\sigma_I) = (\beta_I\beta\sigma_m^2 + \psi v^2)/(\sigma\sigma_I)$. Note that the index dividend adjustment here shows the general case. For the stock itself, $\psi = 1$, and for the market or any other asset uncorrelated with the stock's residual risk, $\psi = 0$, as in the interest rate adjustment.

However, as with regular options, the out-performance incentive option will generally be exercised before it matures so an American model must be used. The true subjective and objective values can be determined most easily by a change of numeraire. In the index numeraire, the out-performance option's payoff upon exercise is $\text{Max}[s_t - x, 0]$ where $s_t \equiv S_t/I_t$ is the price of the stock in the index numeraire, and $x = X/I_0$ is the indexed strike price. Therefore, the out-performance option can be priced as a simple American option with a strike price of x using an index-numeraire risk-neutral subjective evolution of the stock of $ds/s = (\hat{q}_I - \hat{q})dt + \Upsilon\, d\omega_s$. All of the other features which we have already examined, such as vesting, termination, etc., can be handled in this fashion as well for the out-performance option.

If the index used is the market portfolio, then $\upsilon = 0$. In this common case, most indexed options will be less valuable than unindexed options both objectively and subjectively. Sufficient conditions for this to hold are $q_m < r$ and $\beta > \frac{1}{2}$.[18]

Table 11 shows the objective and subjective values of market-indexed out-performance incentive options at issuance. For these parameter values, the market value of an indexed option is 55% as large as the market value of a regular option. The subjective and objective values are more affected. The subjective value of an index option has a value ranging from 37% to 49% of the regular option. The objective values range from 40% to 53% of that of a regular option. This discrepancy is due to the different effects of systematic and nonsystematic risk.

Risk increases the value of an option. For the market price, the type of risk is irrelevant. For the subjective and objective values, nonsystematic risk increases the expected payoff but simultaneously increases the effective discount rate. For the regular option, both systematic and nonsystematic risk increase the volatility and expected payoff of the option. For the indexed option, the systematic risk has little effect on the payoff and only the nonsystematic risk matters.[19] Therefore, virtually all of the risk giving rise to the indexed option's payoff is also subject to the discount-rate-increasing reduction in present value.

8 Repricing of Incentive Options

Repricing involves the lowering of the strike price on incentive options after the stock price has fallen. This is accomplished either by simply altering the strike price on the

[18] Call option values are increasing in the interest rate and volatility. If $\beta > \frac{1}{2}$, then $\Upsilon < \sigma$, and q_m is used in place of the interest rate. For subjective valuation $\hat{q}_m < r$ if $q_m < r$.

[19] The systematic risk affects the payoff of an indexed option only through the small tracking error due to any difference in beta between the stock and the index.

Table 11: **Objective and Subjective Values and Deltas of Out-Performance Incentive Options**

$S = X = 100 \quad T - t = 10 \quad q = 1\% \quad q_m = 1.5\% \quad \sigma = 30\% \quad \upsilon = 20\% \quad \beta = 1$

Values

Value of marketed option = 24.50
European market value of option = 24.18

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	11.73	6.71	3.59	5.75	1.57	0.32	2.80	0.29	0.01
True Subj.	15.30	11.82	9.41	10.95	7.32	5.34	8.40	5.16	3.65
True Obj.	22.12	19.76	17.45	18.99	14.86	11.78	16.25	11.48	8.63

Deltas

Delta of marketed option = 0.60
European market delta of option = 0.59

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Eu. Subj.	0.34	0.22	0.13	0.19	0.06	0.02	0.11	0.01	0.00
True Subj.	0.48	0.45	0.42	0.44	0.41	0.40	0.42	0.39	0.39
True Obj.	0.53	0.46	0.39	0.44	0.32	0.24	0.36	0.24	0.17

Cost per Unit Delta

Marketed option: 40.53
European model: 40.91

	$\alpha = 25\%$			$\alpha = 50\%$			$\alpha = 75\%$		
$\gamma =$	−2	−4	−6	−2	−4	−6	−2	−4	−6
Indexed	45.69	44.20	41.09	43.18	36.36	29.78	38.90	29.08	22.33
Regular	70.25	71.50	70.50	71.02	67.61	61.09	69.33	61.24	50.82

incentive option or by replacing the existing out-of-the-money options with new at-the-money options.

The practice became widespread in the late 1990s[20] and more recently has begun to draw substantial criticism.[21] For example, in October 1998 alone, more than 100 companies announced such repricings or filed plans with the SEC to do so. Starting in 1999, the FASB required that firms repricing their incentive options make future accounting charges against earnings when the stock price subsequently rose.[22] This ruling by the FASB slowed the repricing of incentive options but only for one year. The substantial drop in stock prices in 2000, particularly in the high-tech sector where incentive options are widely used, brought a resurgence of repricing despite the required accounting charge.[23]

The argument in favor of repricing is that deep out-of-the-money options provide little incentive to perform. The argument against repricing is that the employee is given a windfall when it occurs. As well as simply being viewed as unfair, the windfall shields against the previous stock price drop and thereby reduces their ex ante incentives to the extent such repricings might have been anticipated. To circumvent this latter problem, some repricings are constructed as an exchange where the old out-of-the-money options are swapped for fewer new at-the-money options. Usually the values of the swapped options are equated using the Black-Scholes or other option pricing model.[24]

The costs and effects of repricings have invariably been examined using market-based models. For example consider a firm which grants ten-year incentive options at \$100. With $r = 5\%, q = 0,$ and $\sigma = 30\%$, each option will be worth \$52.57 and have a delta of 0.842. Suppose after one year the stock price has fallen to \$80. Each option is now worth \$33.82

[20]A survey of 113 firms by Pricewaterhouse Coopers revealed that 17.1% had repriced some options between 1988 and 1996. By 1998 that had risen to 42.9%.

[21]The criticism can be found in numerous proxy fights over repricing, the most famous of which was probably the State of Wisconsin Investment Board vs. General Data Comm. The SEC ruled in the board's favor and declared repricings were not "ordinary business". Shareholder proposals on repricings can therefore no longer be omitted from proxies. In addition Institutional Shareholder Services recommends a vote against incentive option plans if the company has a history of repricing options or has the express ability to reprice underwater stock options without first securing shareholder approval.

[22]FASB interpretation FIN 44 deems that once the strike price has been changed it can no longer be considered "fixed" at the time of the grant as required for a qualified option. Therefore, once an option has been repriced the difference between the strike price and the price at exercise must be recognized as a compensation expense. The same rule applies if options are canceled and replaced by others with a lower strike within six months before or after the cancellation.

[23]In addition, many companies have devised modified programs to sidestep the FIN 44 ruling. These include: extending the life of the underwater options, granting new options only six months and a day after cancellation of the original, replacing options with stock grants, and granting "paired" options with a strike equal to the current stock price and which expire six months and a day after the stock price reaches the original strike.

[24]Dial and Murphy [1995] describe one of the earliest Black-Scholes repricings implemented by General Dynamics in February 1991.

and the delta has decreased to 0.759. If the option is repriced by replacing it with a new ten-year, at-the-money option, it will be worth \$42.05, and the delta will be restored to 0.842. This is an increase in value of \$8.24 or 24%.[25] Clearly repricing can be costly if there are a substantial number of options.

Now consider this same case evaluated objectively by the firm and subjectively by a manager with $\alpha = 0.25$ and $\gamma = -4$ when $v = 20\%$. The option's subjective value and delta are \$28.82 and 0.573 originally and \$17.44 and 0.476 after the stock price falls. Subsequent to the replacement, the subjective value and delta are \$23.06 and 0.573. Again the subjective delta is restored to its original value, but manager views the windfall as a 32% increase in value — even more than indicated by the market-price model. The objective cost, which was originally \$44.52 per option, increases from \$29.49 to \$35.62 with the repricing.

Therefore, the market-based model concludes that the repricing increases the value and delta of each option by \$8.24 and 0.083 — a cost of \$99.25 per unit of delta. In actuality, the more important subjective delta is increased by 0.097 at an objective cost of \$6.13 — a cost of only \$63.32 per unit of delta. So the cost-benefit of the repricing is not nearly as bad as the market model would suggest. In fact, in these terms, the repricing is cheaper than the original option which had a cost-benefit ratio of 44.52/0.573 = 77.68. The market-based model comes to exactly the opposite conclusion since it's original and repricing cost-benefit ratios are 62.45 and 99.25.

Suppose, instead, a Black-Scholes repricing is used. Model prices of the market values are used to determine the replacement ratio so each existing option will be replaced by 33.82/42.01 = 0.804 new options. The market value per original option will remain at \$33.82 after the repricing but the delta per original option will be reduced to $0.804 \cdot 0.842 = 0.677$.

A value-preserving repricing like this can never be justified based on the same model used in the repricing itself. The value will always remain unchanged, but the delta will drop so incentives will also be lessened.

A Black-Scholes repricing *can* make sense when the subjective delta and objective cost are considered. The 0.804 new options will have a total subjective delta of 0.461. This is smaller than the pre-repriced delta of 0.476, but the objective cost is also lower, \$28.64 rather than \$29.49. The 85 cent savings only gives up 0.015 units of delta. In fact, the cost-benefit ratio of \$54.75 is substantially below that on a one-for-one repricing (\$63.32) and on the original option (\$77.68).

[25] If the strike price is simply lowered to \$80 and the same expiration date is kept, the new value and delta will be \$39.79 and 0.829. The remaining analysis below is similar.

This analysis has been on the ex post effects of repricing. The possibility of repricing also has ex ante effects. Managers recognizing that their options might be repriced after a fall in the stock price will understand that the value and delta of the option will be miscalculated by a model that ignores repricing. Brenner, Sundaram, and Yermack [2000] and Johnson and Tian [2000] have studied this problem using a market based model. The calculated value of the option will understate (overstate) the actual value of the option if the contract received in the repricing is more (less) valuable than that given up. Similarly the delta will be overstated (understated). The former statement is obvious; the latter follows from the first since the understatement in value is largest when the stock price is lowest and repricing is more apt to occur. The size of the effect will depend on the type of repricing and likelihood that it will occur under various conditions. The subjective and objective values will similarly be affected.

For example, suppose it is known that the option will be repriced by replacing it with a new at-the-money option when price falls to L.[26] Then the pre-repriced value of the contract is the solution to the market or subjective pricing equation with the usual maturity and early-exercise conditions. To handle repricing we apply

$$\hat{C}(L,t) = \hat{\mathcal{C}}(L,t;t+T-t_0;L) \tag{45}$$

For the ex ante objective or market values we use the ex post objective or market value on the right hand side.

The value of the option before it is repriced is

$$\begin{aligned} C = \mathcal{S}(S,t;T;\{S_{\min} > L\}\,\&\{S_T > X\}) - X\mathcal{D}(S,t;T;\{S_{\min} > L\}\,\&\{S_T > X\}) \\ + \mathcal{C}(L,0;\hat{t}+T;L)\mathcal{T}(S,t;T;L)\ . \end{aligned} \tag{46}$$

For a given L, the value $\mathcal{C}(\cdot)$ is a constant. It can be determined before the time the repricing occurs and is a constant.

If early exercise must be considered because the stock is paying dividends or a subjective or objective value is being determined, then repricing occurs at L and exercise occurs

[26] Usually it is not known exactly when a repricing will occur. In this case we can estimate the option value and delta by taking an average across the possible repricing barriers using the risk-neutral probability of each.

at an upper boundary. Again using a constant exercise policy approximation, the value of the option is

$$
\begin{aligned}
C \gtrapprox\, & \mathcal{S}(S,t;T;\{S_{\min} > L\}\,\&\{S_{\max} < k\}\,\&\{S_T > X\}) \\
& - X\mathcal{D}(S,t;T;\{S_{\min} > L\}\,\&\{S_{\max} < k\}\,\&\{S_T > X\}) \\
& \quad + \mathcal{C}(L,0;\theta;L)\mathcal{T}(S,t;T;L;\{\tau_L < \tau_k\}) + (k-X)\mathcal{T}(S,t;T;k;\{\tau_k < \tau_L\})\ .
\end{aligned}
\tag{47}
$$

The first two terms give the present value of the payoff of $S_T - X$ at expiration if the option is in-the-money and has not been repriced or exercised. The third term gives the present value of the repriced option if repricing occurs. The final term gives the present value of early exercise. The variables τ_L and τ_k are first-passage times — the times when the stock price first reaches L or k. The option is repriced only if it has not been exercised, $\tau_L < \tau_k$; it is exercised at k only if it has not been repriced, $\tau_k < \tau_L$. The option may also be exercised after it is repriced though this would be at a price lower than k. This exercising will be reflected in the value $\hat{\mathcal{C}}(L,0;t+T;L)$. The value of the digitals are

$$
\begin{aligned}
&\mathcal{S}(S,t;T;\{S_{\min} > L\}\,\&\{S_{\max} < k\}\,\&\{S_T > X\}) \\
&\quad = Se^{-q(T-t)} \sum_{n=-\infty}^{\infty} (k/L)^{2n(\lambda+1)} \left[\Phi(H(XLA_n/k)) - \Phi(H(LA_n))\right] \\
&\qquad\qquad - (k^{1-n}L^n/S)^{2\lambda+2}\left[\Phi(H(S^2X/LkA_n)) - \Phi(H(S^2/LA_n))\right] \\
&\mathcal{D}(S,t;T;\{S_{\min} > L\}\,\&\{S_{\max} < k\}\,\&\{S_T > X\}) \\
&\quad = e^{-r(T-t)} \sum_{n=-\infty}^{\infty} (k/L)^{2n\lambda} \left[\Phi(h(XLA_n/k)) - \Phi(h(LA_n))\right] \\
&\qquad\qquad - (k^{1-n}L^n/S)^{2\lambda}\left[\Phi(h(S^2X/LkA_n)) - \Phi(h(S^2/LA_n))\right] \\
&\mathcal{T}(S,t;T;k;\{\tau_k < \tau_L\}) = \left(\frac{k}{S}\right)^{\lambda} \sum_{n=1}^{\infty}\Bigg[\left(\frac{SA_n}{L}\right)^{-\lambda}\Phi(h_\kappa(L/A_n)) + \left(\frac{SA_n}{L}\right)^{\lambda}\Phi(H_\kappa(L/A_n)) \\
&\qquad\qquad - \left(\frac{S}{LA_n}\right)^{\lambda}\Phi(-H_\kappa(LA_n)) - \left(\frac{S}{LA_n}\right)^{-\lambda}\Phi(-h_\kappa(LA_n))\Bigg] \\
&\mathcal{T}(S,t;T;L;\{\tau_L < \tau_k\}) = \left(\frac{L}{S}\right)^{\lambda} \sum_{n=1}^{\infty}\Bigg[\left(\frac{S}{kA_n}\right)^{\lambda}\Phi(-H_\kappa(kA_n)) + \left(\frac{S}{kA_n}\right)^{-\lambda}\Phi(-h_\kappa(kA_n)) \\
&\qquad\qquad - \left(\frac{SA_n}{k}\right)^{-\lambda}\Phi(h_\kappa(k/A_n)) - \left(\frac{SA_n}{k}\right)^{\lambda}\Phi(H_\kappa(kA_n))\Bigg] \\
&\quad \text{where} \qquad A_n \equiv L^{2n-1}/k^{2n-1}
\end{aligned}
\tag{48}
$$

Similar analysis can be applied to other repricing problems. For example suppose any repriced option may itself be repriced. The value of an option subject to repricing at a series of stock prices, $L_1, L_2, \ldots$, can be valued using this same method recursively. The greater the number of possible repricings, the higher is the ex ante option value and the lower is its delta.

If the strike price is reset to any other fixed multiple of the prevailing stock price or if there is an n for one repricing, the same method is used. At repricing, the option would be worth $n\hat{\mathcal{C}}(L, t; t + T - t_0; aL)$ for known values of n and a so this value is used on the right-hand side of (45).

For a Black-Scholes repricing, the right-hand-side of (45) is $[\mathcal{C}(L, t; T; X)/\mathcal{C}(L, t; T + t - t_0; L)]\hat{\mathcal{C}}(L, t; t + T - t_0; L)$. In this fraction the option prices are always the (modeled) market values, never the objective of subjective values. This problem can be handled in the same fashion with a time-dependent payoff at first touch. See Ingersoll [2000].

9 Conclusion

Incentive options are an important component of compensation. Understanding the true cost and incentive effects of these options is likewise important. Recently companies have been required to estimate and report the cost of granted options. Usually this is done by using the Black-Scholes or binomial model or simple modifications thereof. This paper shows that these models my substantially misstate the cost and incentive effects of such options.

A model was developed which allows estimating the costs of such options. The same model can be used to determine the market value, the subjective value and the objective value. The latter value, which is actual the cost to the firm of issuing the option is particularly important and has usually be completely neglected — even in analyses which look at subjective values.

The model here is no more difficult to use than is the Black-Scholes model. In fact, it is simply the Black-Scholes model with modified parameters. Since the model is based on the Black-Scholes model, it can easily be extended to handle all of the modifications seen in incentive options. Vesting, employment termination, indexing, and repricing were discussed here. Other problems can also be tackled.[27]

[27] See Ingersoll [2001] for example for a treatment of "reload" options.

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