Non-Monotonicity of the Tversky-Kahneman Probability-Weighting Function A Cautionary Note

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Cumulative Prospect Theory (CPT) has been used in a wide variety of economic models as an alternative to Expected Utility Theory (EUT). It is used to account for a number of anomalies in the observed behavior of economic agents. Like EUT, CPT uses a utility or value function to rate any given outcome. In CPT, however, the utility function is loss averse (S-shaped) rather than risk averse (concave). The other difference between the two theories is the use of decision weights rather than probabilities in computing the expectation.

This note provides a caution on the use of a popular form of decision weights — those proposed by Tversky and Kahneman (1992) in their introduction of the theory. In particular, the weights generated by their proposed probability weighting function can be negative for some parameter values. This could lead to evaluations that chose first-order stochastically dominated gambles rather than the dominating one.

The parameter values that can cause such problems are small relative to estimated values. Nevertheless, failure to recognize this problem may prevent the proofs of some theorems or allow the proof of some theorems which would be invalid with properly restricted probability weights.

The Tversky-Kahneman Probability-Weighting Function

In Prospect Theory, the "expected" utility of any risky prospect with outcomes x_i is computed as

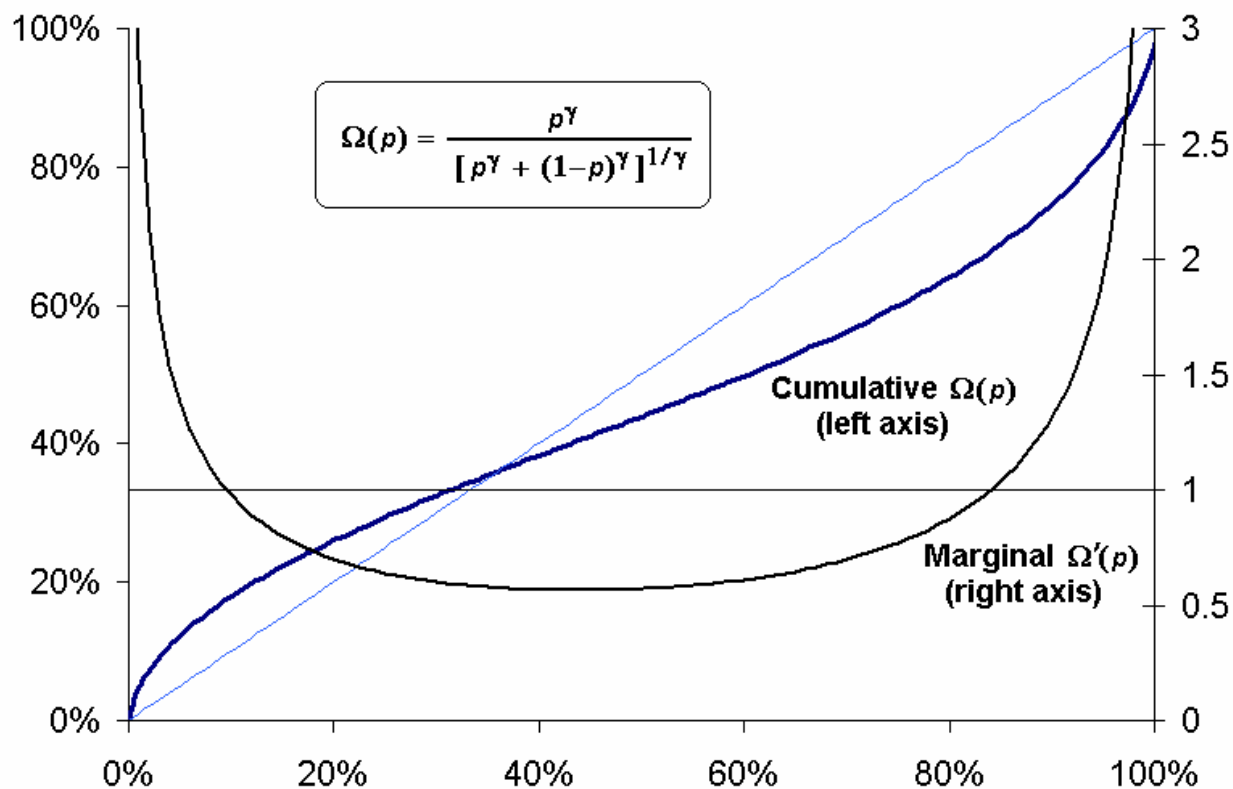$$\sum v(x_i)\omega_i(\mathbf{p}) \tag{1}$$

where v is a value (utility) function and ω_i are a set of decision weights. In the original application of Prospect Theory, each decision weight was a function of the probability of the outcome; in CPT, decisions weights depend on the probability distribution. They are determined as first differences of a probability-weighting function applied to the cumulative probabilities, p_i, $\omega_i = \Omega(p_i) - \Omega(p_{i-1})$.[1]

The particular probability-weighting function originally proposed by Tversky and Kahneman (1992) is

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[1] In CPT, separate weighting functions are used for the probability distribution of losses and the complementary distribution of gains, but this is irrelevant to the issue presented here which is about the properties of the weighting function itself. If $p(x)$ is a cumulative probability function for a continuous distribution with density $p'(x)$, then the continuous decision weighting density function is $\omega(x) = \Omega'(p)p'(x)$.

Probability Weighting Function

$$\Omega(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{1/\gamma}}$$

Cumulative $\Omega(p)$ (left axis)

Marginal $\Omega'(p)$ (right axis)

$$\Omega(p) = \frac{p^{\gamma}}{[p^{\gamma} + (1-p)^{\gamma}]^{1/\gamma}} \qquad\qquad 0 < \delta \leq 1 \qquad\qquad (2)$$

where p is the cumulative probability of the distribution of gains or losses. As shown in Figure 1, this weighting function has an inverted S-shape and is subproportional.[2] The latter property allows for the Allais paradox and can explain the preference for lottery tickets.

Since the decision weights take the place of probabilities in the computation of "expected" utility, they must be positive. If they are not positive, then a first-order stochastically dominated gamble might be preferred to the dominating gamble.[3] The decision weights are first differences and will be positive for all possible risky prospects only if the weighting function is strictly increasing. But the Tversky-Kahneman probability-weighting function does not posses this property for all parameter values.

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[2] A function Ω is subproportional if for $p > q$ and $0 < \lambda < 1$, $\Omega(\lambda q)/\Omega(q) > \Omega(\lambda p)/\Omega(p)$.

[3] Let $x > x'$ and consider two risky prospects which are identical except for the inclusion of either x or x' as the ith outcome. Now suppose that $\omega_i < 0$, then the dominated gamble including x' will be preferred.

Let $F(p) \equiv \ell n[\Omega(p)]$. This is a strictly-increasing transform of the probability weighting function so Ω is strictly increasing if and only if F is. The derivative of $F(p)$ is

$$F'(p) = \frac{\gamma}{p} - \frac{p^{\gamma-1} - (1-p)^{\gamma-1}}{p^{\gamma} + (1-p)^{\gamma}} = \frac{\gamma p^{\gamma} + \gamma(1-p)^{\gamma} - p^{\gamma} + p(1-p)^{\gamma-1}}{p[p^{\gamma} + (1-p)^{\gamma}]}. \tag{3}$$

Since the denominator is positive, F' is negative whenever the numerator is negative. Dividing the numerator by $(1-p)^{\gamma}$, we see that F' is negative when

$$(\gamma-1)\frac{p^{\gamma}}{(1-p)^{\gamma}} + \gamma + \frac{p}{1-p} < 0. \tag{4}$$

Define $x \equiv p/(1-p)$, then we are looking for values of the weighting function parameter, γ, for which the function

$$f(x) \equiv (\gamma-1)x^{\gamma} + x + \gamma \tag{5}$$

can take on negative values for some positive x. Note that $f(0) = \gamma > 0$, $f(\infty) = \infty$, and f is strictly convex

$$f'(x) = \gamma(\gamma-1)x^{\gamma-1} + 1 \gtrless 0$$
$$f''(x) = \gamma(\gamma-1)^{2}x^{\gamma-2} > 0. \tag{6}$$

Therefore, the function f will take on negative values over some range when it is negative at its minimum. The minimum occurs at

$$f'(x) = \gamma(\gamma-1)x^{\gamma-1} + 1 = 0 \quad \Rightarrow \quad x_{\min} = [\gamma(1-\gamma)]^{1/(1-\gamma)} \tag{7}$$

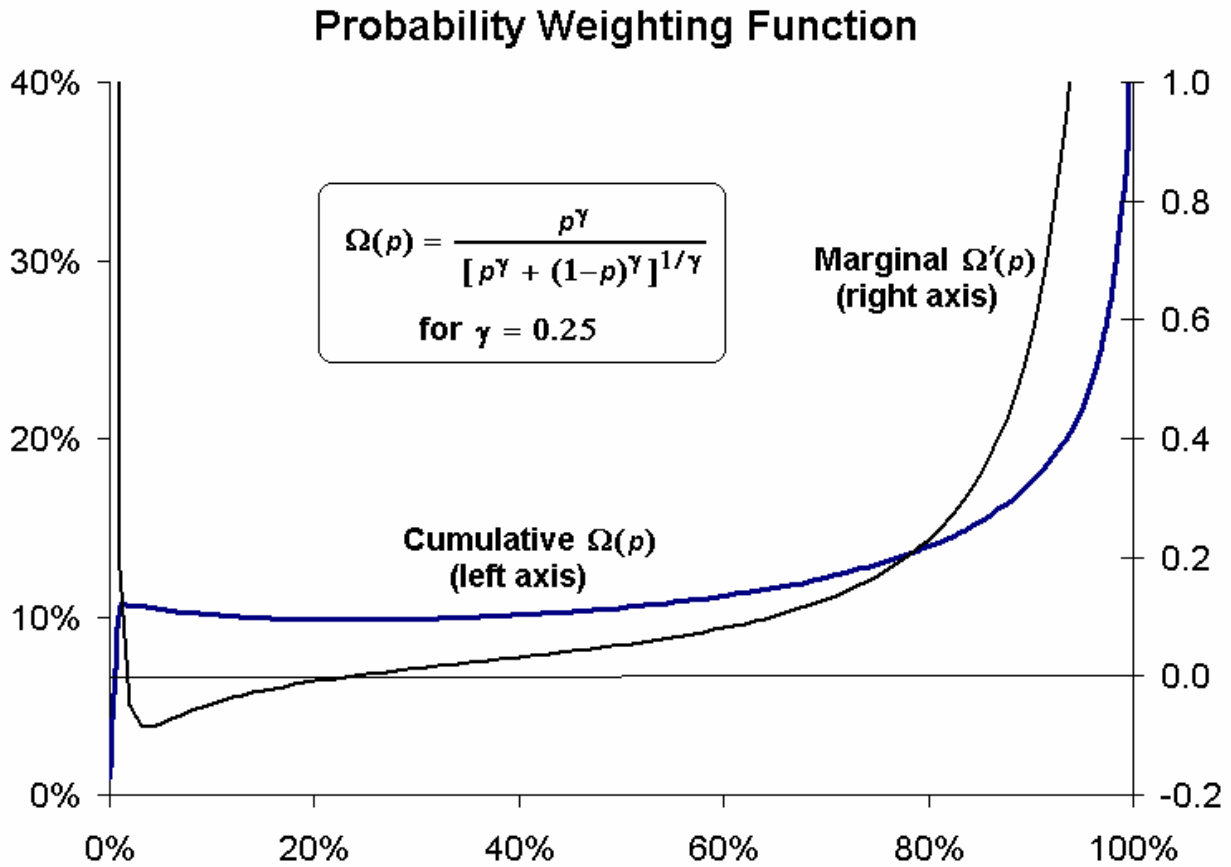So the minimum value of f for a given value of the parameter γ is

$$f_{\min}(\gamma) \equiv f(x_{\min}) = (\gamma-1)[\gamma(1-\gamma)]^{\gamma/(1-\gamma)} + [\gamma(1-\gamma)]^{1/(1-\gamma)} + \gamma$$
$$= \gamma\left[1 - (1-\gamma)^{(2-\gamma)/(1-\gamma)}\gamma^{(2\gamma-1)/(1-\gamma)}\right]. \tag{8}$$

By inspection, there is a critical value, γ_{critical}, where $f_{\min} = 0$. For all values of γ less than this critical value, $f_{\min} < 0$, and the probability weighting-function is therefore nonmonotonic. The critical value satisfies

$$1 = (1-\gamma)^{(2-\gamma)}\gamma^{(2\gamma-1)} \quad \Rightarrow \quad \gamma_{\text{critical}} \approx 0.279. \tag{9}$$

Figure 2 illustrates the Tversky-Kahneman probability-weighting function for a parameter value of $\gamma = 0.25$. Note that the derivative is negative over the approximate range $1.6\% < p < 23.6\%$. Any continuous probability density will have a corresponding decision weight density which is negative over this range. A discrete distribution with two (or more) cumulative probabilities in this range will have a negative decision weight(s) for the corresponding outcome(s), and could have

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others. If just one cumulative probability is in this range, the corresponding outcome may have a positive or negative weight.[4]

Probability Weighting Function

$$\Omega(p) = \frac{p^\gamma}{[\,p^\gamma + (1-p)^\gamma\,]^{1/\gamma}}$$

$$\text{for } \gamma = 0.25$$

Marginal $\Omega'(p)$ (right axis)

Cumulative $\Omega(p)$ (left axis)

Tversky and Kahneman (1992) estimated the parameters $\gamma = 0.69$ for losses and $\gamma = 0.61$ for gains. Other estimates have varied widely; Camerer and Ho (1994) report on eleven tests whose estimates for γ range from 0.28 to 1.87 with an average value of 0.55. All of these point estimates for γ are above the critical value and would lead to a monotone probability-weighting function and generate only positive decision weights. Nevertheless, the range of estimates reported does not engender great confidence that all future studies would continue to do this.

Probably of more concern, however, is the use of this probability-weighting function in theory. If this behavior is posited, and no restriction is placed on the parameter, it is very possible, that some results might not be provable due to the possibility of first-order stochastic dominance. Conversely some results might be obtained that could not hold if the weighting function did not generate negative decision weights.

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[4] Precisely if the cumulative probability, p_i, for a discrete distribution lies in the range of decreasing Ω, then either or both ω_i and ω_{i+1} could be negative. If p_{i-1} (p_{i+1}) is also in the decreasing range, then ω_i (ω_{i+1}) will definitely be negative.

Other Probability-Weighting Functions

Several other probability weighting functions have also been proposed. Some of them are:

$$\Omega(p) = \frac{p^{\gamma}}{[p^{\gamma} + (1-p)^{\gamma}]^{\delta}} \qquad \text{Wu and Gonzalez (1996)}$$

$$\Omega(p) = \frac{\delta p^{\gamma}}{\delta p^{\gamma} + (1-p)^{\gamma}} \qquad \text{Lattimore, Baker, and Witte (1992)} \qquad (10)$$

$$\Omega(p) = \gamma \exp\left[-\beta(-\ell n\, p)^{\alpha}\right] \qquad \text{Prelec (1998)}$$

In each case, $0 < \gamma \le 1$, and the other parameters are positive.

The LBW and Prelec probability weighting functions are strictly increasing for all relevant parameter values

$$\frac{d}{dp}[\ell n\, \Omega(p)] = \begin{cases} \dfrac{\gamma(1-p)^{\gamma-1}}{\delta p^{\gamma+1} + p(1-p)^{\gamma}} > 0 & \text{Lattimore, Baker, and Witte (1992)} \\ \\ \dfrac{\alpha\beta}{p}(-\ell n\, p)^{\alpha-1} > 0 & \text{Prelec (1998)} \end{cases} \qquad (11)$$

They cannot, therefore, assign negative decision weights to any outcomes for any probability distribution.

If $\delta \le 1$, the Wu and Gonzalez weighting function is strictly increasing and also cannot assign negative decision weights.[5] However, for $\delta > 1$, negative decision weights are possible in some cases. This is clearly true since the TK weighting function is a special case, $\delta = 1/\gamma$.

Conclusion

The purpose of this note is not to comment directly on Cumulative Prospect Theory. Rather it is to point out a possible obstacle in its use. Some parameter values of the commonly used probability-weighting function, originally proposed by Tversy and Kahneman, can lead to negative decision weights and the preference for first-order stochastically dominated prospects. This could lead to awkward interpretations of results. In addition certain consequences might be found which would not obtain were only positive decision weights allowed, or conversely "general" propositions might remain unverifiable.

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[5] This is true for Karmarkar's (1978) weighting function where $\delta = 1$.

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