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A NEW APPROACH OF VALUING ILLIQUID ASSET PORTFOLIOS

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Abstract:

This paper proposes a new approach of valuing portfolios that contain illiquid assets. The approach has three major advantages. First, the estimators are arithmetic averages of individual asset returns or their proxies, so they strictly correspond to actual portfolio returns. Second, the approach is able to value portfolios in which assets are arbitrarily weighted, including equal-weighted, price-weighted, and value-weighted portfolios. Third, the model is easy to extend to incorporate asset characteristic data to improve the accuracy. Simulations with actual data of Dow Jones Industrials show that this new approach provides superior estimators than some currently available alternatives.

A New Approach of Valuing Illiquid Asset Portfolios

Many important assets transact infrequently. For example, the real estate, art, and bond markets are generally considered illiquid. In real estate and art markets, assets tend to be held for years or even decades between sales. In United States bond markets, less than 10% of bonds transact daily. At the same time, while more high-frequency data of stock trades and quotes are available, researchers encounter infrequent trading problem more often, for example when calculating high frequency stock price index. The global equity market, if considered as an integrated market, is also illiquid: while the global equity market is considered open, regional exchanges may be closed thus their listed equities may not be tradable. Not only many existing asset markets are illiquid; many to-be-established markets might be so. For example, for the new macro markets originally proposed by Shiller (1993a), such as national income and labor income markets, the underlying cash market prices may be observable only infrequently. Clearly illiquid assets are widely spread within the economy, which therefore raises the question how to value portfolios that contain illiquid assets, in spite of a potential paucity of transaction data.

One well-known method for estimating the returns of illiquid asset portfolios is Repeat sales regression (RSR). This technique estimates the time series returns using the observed transaction prices for a subset of assets. First suggested by Bailey, Muth, and Nourse (1963), the RSR has been the subject of a great deal of discussion since then.[1] The original RSR model has two serious limitations. First, its estimators are geometric averages of cross-section

[1] For example, Case and Shiller (1987, 1989), Clapp and Giaccotto (1992, 1999), Geltner (1997), Goetzmann (1989, 1992), Goetzmann and Peng (2000), Goetzmann and Spiegel (1995, 1997), Shiller (1991, 1993b). All suggest various ways to modify and improve the technique.

individual asset returns, while the true returns for a portfolio, no matter an equal-weighted or a value-weighted one, are always arithmetic averages of individual asset returns. Jensen's inequality implies that the geometric average of any set of positive numbers not all equal is less than the arithmetic average of them. Thus the RSR estimators tend to be biased away from actual portfolio returns. Even when all transactions are observed and the actual portfolio returns are already known, the RSR estimators still don't equal the actual returns. Goetzmann (1992) proposes a correction method that approximates the arithmetic averages given the geometric averages, under the assumption that the asset returns in each period are identically lognormally distributed. This method works well in simulations. However, it needs to estimate unobserved cross-sectional variances, which may not be easy in some scenarios such as when time series data are heteroskedastic. As an alternative, Goetzmann and Peng (2000) propose a method that directly provides arithmetic average estimators of the equal-weighted portfolio returns. The second limitation of the original RSR method is that it actually provides estimators for equal-weighted portfolio returns only, while one may be more interested in price-weighted, value-weighted, or other special-weighted portfolios. Shiller (1991) proposes estimators, either price-weighted or equal-weighted, that are analogous to the original RSR estimators but are arithmetic averages, which is called arithmetic repeat sales estimator (ARS). However, more flexible approaches that are able to value arbitrary-weighted portfolios would be desirable.

Researchers have proposed methods using both transaction data and data of asset characteristic to estimate returns of infrequent-traded asset portfolios. For example, Case and Quigley (1991), Case *et al* (1991), Clapp and Giaccotto (1992), “hedonic repeated measures” method (HRM) by Shiller (1993b), and “distance-weighted repeat-sales” procedure (DWRS) by Goetzmann and Spiegel (1997). The primary methodological advantage of using both transaction and characteristic data lies in its ability to exploit the relation between asset returns with its characteristics. Limitations of these methods are that they may not provide return estimators for arbitrarily weighted portfolios, and their estimators may not have natural interpretations, such as being arithmetic means of individual asset returns.

This paper proposes a new approach of valuing portfolios that contain illiquid assets based on the method of moment, which is called the GMM approach throughout. The GMM approach has three major advantages. First, it is capable of valuing arbitrarily weighted portfolios as long as asset weights are known or derivable. Few previous methods claim to be able to do so. Second, all GMM estimators of portfolio returns are cross-sectional arithmetic averages of individual asset returns (or proxies of them), so the estimators strictly correspond to actual portfolio and no correction is needed, which is an important improvement over the currently broadly used RSR method. Third, the GMM approach is potentially extendable to incorporate asset characteristic data to improve the accuracy. On one hand, the characteristic data help to differentiate one asset from another, which facilitate the correction for the biased sample problem that transactions more likely take place upon a subset of assets in the portfolio (this will be shown in section four). On the other hand, since assets with different characteristics may have different return processes, characteristic data would help to proxy individual asset's returns more accurately. Therefore the GMM estimators of portfolio returns would be more accurate since they are averages of proxies of individual asset returns.

This paper notices a finite sample problem of the new approach that infrequently traded assets tend to be over-weighted in the estimation. A correction method is proposed to provide the finite sample version of the GMM approach. To compare its performance with that of the RSR and the ARS and a simple method that estimates a portfolio's return for a period by averaging all available individual returns for that period, actual financial data are used to do simulations. Each simulation firstly constructs infrequently transaction data set by drawing some of actual daily prices for the Dow Jones Industrials Index (DJII) stocks over September to December 1999, then estimates the actual DJII daily returns with the limited data set. The accuracy of each method is measured with four different statistics. They are the squared error of the geometric mean of return estimators, the standard deviation of the return estimators, the R^2 resulting from regression of the actual DJII daily returns upon the estimated time series returns, and the mean squared differences between the actual and the estimated returns. The simulations show that the GMM approach is superior to other methods on measuring the overall performance of portfolios and on capturing period to period return evolution as well. The superiority of the GMM is more obvious on valuing portfolios containing both liquid and illiquid assets.

The paper is organized as follows. Section 1 presents the mathematical model of return process. Section 2 discusses the basic model estimation and illustrates the estimators by estimating an extremely small data set. It also discusses the finite sample problem and proposes a correction method. Section 3 describes the procedure of the simulation test and reports results. Section 4 discusses potential extensions of the model, including incorporating characteristic data into estimation and using them to correct the biased sample problem. Section 5 concludes. An appendix presents details of estimation algorithms for the equal-weighted and the price-weighted portfolios.

I. Mathematical Model of Return Process

I.1. Asset returns

Define capital appreciation of asset a in time period t, $r_{a,t}$, as the ratio of the price of the asset at the end of time period t over its price at the end of period $t-1$.

$$r_{a,t} \equiv P_{a,t} / P_{a,t-1} .$$

Assume that the $r_{a,t}$ is determined as following:

$$r_{a,t} = E(r_{a,t} | m_t, c_{a,t}) \boldsymbol{e}_{a,t} . \tag{1}$$

The term m_t is a set of portfolio-wide common factors that affect all assets' returns in time period t. The term $c_{a,t}$ is a set of characteristics of asset a in time t. The error term $\boldsymbol{e}_{a,t}$ captures asset-specific events that are responsible for unexpected change of price. Assume $E(\boldsymbol{e}_{a,t} | m_t, c_{a,t}) = 1$ and $\boldsymbol{e}_{a,t}$ is independent.

Based on these assumptions, common factors and asset characteristics jointly determine an asset's expected return in a time period. The common factors could be macroeconomic variables like the risk-free interest rate, inflation rate, unemployment rate, and so on. For houses, asset characteristics could be hedonic variables such as location or square-feet of floor space. For equities, they could be P/E ratio, B/M ratio, capitalization and so on. For bonds, they could be bond maturity, rating, coupon rate or other characteristics. The assumptions about the asset return process are consistent with that assets with different characteristics may have different return processes.

I.2. Portfolio returns

A portfolio consists of units of value, say dollars, that are invested in different assets. The return of a portfolio in time period t, r_t, equals the ratio of the portfolio value at the end of time period t over its value at the end of time period $t-1$. Suppose a portfolio consists of N dollars invested in A different assets at the end of time period $t-1$, and a dollar d becomes $r_{d,t}$ at the end of time t. Then the return of this portfolio in time period t is

$$r_t \equiv \frac{\sum_{d=1}^{N} r_{d,t}}{N}. \tag{2}$$

Since the return of a dollar equals the return of the asset in which this dollar is invested, all dollars invested in the same asset have same value at the end of period t. Thus the portfolio return in time t can be expressed as average of asset returns.

$$r_t = \sum_{a=1}^{A} \left(w_{a,t} r_{a,t} \right). \tag{3}$$

The term $w_{a,t}$ is the *weight* of asset a in this portfolio. It equals the proportion of the dollar value of the portfolio invested in asset a at the end of $t-1$.

I.3. Return of a random dollar in a portfolio

Dollars in a portfolio are distinguished from each other by the characteristics of assets in which they are invested. The probability for a randomly selected dollar to have specific characteristics equals the weight of the asset having these characteristics in the portfolio. Obviously differently weighted portfolios have different probability distributions of dollar characteristics. For example, consider portfolios of two assets: one risky bond and one risk-free bond. For the equal-weighted portfolio, the probability of a random dollar being risk-free is 0.5. For a price-weighted portfolio, if the risk-free bond has higher price, the probability of a random dollar being risk-free is larger than 0.5.

Let $f_t(c)$ denote the probability of a randomly selected dollar in time period t having characteristic set c. Denote by $\boldsymbol{g}_t$ the expected return of the random dollar in period t conditional upon the set of common factors, then

$$\boldsymbol{g}_t \equiv E(r_{d,t}|m_t) = E(E(r_{d,t}|m_t, c)) = \sum_c (E(r_{d,t}|m_t, c) f_t(c)) = \sum_{a=1}^{A} E(r_{a,t}|m_t, c_{a,t}) w_{a,t} , \qquad (4)$$

which equals the expected portfolio return. The second equality holds because of the law of iterated expectation. The last equality holds since the probability for the random dollar to have specific characteristics equals the weight of the asset in the portfolio that has these characteristics. Using equation (1) and (4) and the fact that a dollar's return equals the return of the asset in which this dollar is invested, one can always write the return of a randomly selected dollar d in the portfolio as

$$r_{d,t} = \boldsymbol{g}_t \boldsymbol{h}_{d,t} \boldsymbol{e}_{d,t} \qquad (5)$$

with $\boldsymbol{h}_{d,t} \equiv E(r_{d,t}|m_t, c_{d,t}) / E(r_{d,t}|m_t)$, which is a function of the dollar’s characteristics. Since the dollar is randomly selected, it has random characteristics and the term $\boldsymbol{h}_{d,t}$ is a random scalar with $E(\boldsymbol{h}_{d,t}|\boldsymbol{g}_t) = 1$ by the law of iterated expectation.

Equation (5) connects the return of a randomly selected dollar with the expected return of the portfolio. It has an intuitive interpretation. The return of a randomly selected dollar in a portfolio consists of three parts. The first part is the expected return of the portfolio. The second part is the expected deviation from the portfolio's expected return due to the asset's characteristics. The third part is a random shock.

II. Model Estimation

II.1. GMM estimators

Now assume asset characteristics are not observable and the data consist of transaction prices and time. Assume that a transaction always take place at the end of a time period. A repeat-sale observation consists of the first transaction price, the time of the first transaction, the second transaction price, and the time of second transaction. For observation *n*, denote by B_n the first transaction price (the purchase or buy price), by S_n the second transaction price (the sale price), by b_n the time of first transaction, by s_n that of the second transaction. The holding interval of observation *n*, denoted by H_n, consists of all time periods later than b_n and not later than s_n, i.e., $H_n \equiv \{t | b_n + 1 \le t \le s_n\}$. The length of H_n is denoted by T_n, so $T_n \equiv s_n - b_n$. Assume there are N repeat-sale observations and $T+1$ time periods in the sample, numbered from 0 to *T*. For time period *t*, denote by $O_t \equiv \{n | t \in H_n\}$ the set of all observations that have this time period in their holding intervals. Define the size of O_t, i.e. the number of observations that belong to O_t, by N_t.

Let y_n equal the compound return from the observed buy to sell,

$$y_n \equiv S_n / B_n .$$

Then,

$$y_n = \prod_{t \in H_n} r_{n,t} = \prod_{t \in H_n} g_t \prod_{t \in H_n} (h_{n,t} e_{n,t})$$

$$y_n / \prod_{t \in H_n} g_t = \prod_{t \in H_n} (h_{n,t} e_{n,t}) .$$

Since $E\left[\prod_{t \in H_n} (h_{n,t} e_{n,t})\right] = 1$, the moment conditions $E(y_n / \prod_{t \in H_n} g_t - 1) = 0$ for $n = 1,...,N$, yield a parameter-defining mapping under suitable regularity conditions, which are assumed to hold.

Sample counterparts to the moment conditions define the estimator of g:

$$\sum_{n \in O_t} w_n \left(y_n / \prod_{s \in H_n} \hat{g}_s - 1 \right) = 0 \text{, for } t = 1,...,T . \tag{6}$$

The term $w_{n,t}$ indicates how many dollar samples the repeat-sale observation n provides, so it equals the weight of the asset that corresponds to the repeat-sale observation n. An observation provides more dollar samples of the portfolio if its asset has heavier weight in the portfolio. Thus by choosing different $w_{n,t}$, one could use the same data set to estimate returns of different portfolios. For example, each observation provides the same amount of dollar sample for the equal-weighted portfolio, so $w_{i,t}$ should equal to each other to estimate the returns of the equal-weighted portfolio. At the same time, an observation provides the amount of dollar sample proportional to its asset price for the price-weighted portfolio, so $w_{i,t}$ should be proportional to its asset price to estimate the price-weighted portfolio returns.

Rearranging equation (6), the estimator of portfolio return in time period t is

$$\hat{g}_t = \sum_{n \in O_t} w_{n,t} \left(y_n / \prod_{\{s / s \in H_n, s \neq t\}} \hat{g}_s \right). \tag{7}$$

Since the y_n term is a compound return, the $y_n / \prod_{\{s/s\in H_n, s\neq t\}} \hat{g}_s$ term is the compound return from which all expected returns for time periods within holding interval except t are subtracted. Thus this term is a proxy of an asset's return in time period t. Then obviously the estimator $\hat{g}_t$ is an arithmetic average of returns (or proxies of returns) of individual assets in time period t. It includes the returns (or proxies of returns) of all assets in the portfolio, as long as the assets are traded at least once before and once after current time period. Also, the return of each asset, no matter the asset is traded frequently or infrequently, is directly included in the estimator only once. Thus, this estimator doesn't directly over count frequently traded or infrequently traded assets.

A very nice property of the estimators is that: when all assets are frequently traded, the estimators are averages of individual asset returns, which exactly equal the actual portfolio returns.

$$\hat{g}_t = \sum_{n\in O_t} w_{n,t} y_n = \sum_{n\in O_t} w_{n,t} \left(S_n / B_n\right).$$

II.2. Returns of Equal-weighted and Price-weighted Portfolios

From equation (6), this approach is capable of valuing arbitrarily weighted portfolios, as long as asset weights are known or derivable. The equal-weighted and the price-weighted portfolios may be the most widely used portfolios in research. (Here the value-weighted portfolio is considered as a special case of price-weighted portfolio, in the sense that the asset price in a value-weighted portfolio is price for the whole asset instead of for just one share of the asset.) Here the details of valuing these two kinds of portfolios are presented.

The estimators of the equal-weighted portfolio returns can be easily obtained by letting $w_{n,t} = 1/N_t$, where N_t is defined earlier, as the number of observations that includes time period t in their holding intervals. Denote by g_t^e the return of equal-weighted portfolio in time period t. The estimator-defining equations are

$$\hat{g}_t^e = \frac{1}{N_t}\sum_{n\in O_t}\left(y_n \Big/ \prod_{\{s/s\in H_n, s\neq t\}} \hat{g}_s^e \right), \text{ for } t = 1,\ldots,T . \tag{8}$$

For the price-weighted portfolio, an asset's weight in time period t is proportional to its price at the end of time period $t-1$. Denote by g_t^p the return of price-weighted portfolio in time period t. For illiquid assets, prices are not observable for all time periods, nor are corresponding weights. However, the model itself provides estimators for all unobserved prices. For the asset corresponding to the repeat-sale observation n, an estimator for its price at the end of time period t-1 is $\hat{P}_{n,t-1} = B_n \prod_{s=b_n+1}^{t-1} \hat{g}_s^p$. Then an estimator of the weight of asset n in time period t is

$$\hat{w}_{n,t} = \frac{\hat{P}_{n,t-1}}{\sum_{i\in O_t} \hat{P}_{i,t-1}} = \frac{B_n \prod_{s=b_n+1}^{t-1} \hat{g}_s^p}{\sum_{n\in O_t}\left(B_n \prod_{s=b_n+1}^{t-1} \hat{g}_s^p \right)}.$$

With the estimated weights, the return estimators of the price-weighted portfolio are defined as

$$\sum_{n\in O_t}\left(B_n \prod_{s=b_n+1}^{t-1} \hat{g}_s^p \right) = \sum_{n\in O_t}\left(S_n \Big/ \prod_{s=t}^{s_n} \hat{g}_s^p \right), \text{ for } t = 1,\ldots,T . \tag{9}$$

Rearrange equation (9), the estimator of the price-weighted portfolio return in time period t is

$$\hat{g}_t^p = \frac{\sum_{n \in O_t}\left(S_n / \prod_{s=t+1}^{s_n} \hat{g}_s^p \right)}{\sum_{n \in O_t}\left(B_n \prod_{s=b_n}^{t-1} \hat{g}_s^p \right)}. \tag{10}$$

II.3. A Illustration of Estimators

For an example of the estimators of equal-weighted and price-weighted portfolios, consider a very small data set consisting of two assets and three time periods numbered from 0 to 2. The first asset was sold at the end of each time period, while the second one was sold only at the end of the time period 0 and time period 2. Denote by $P_{1,0}, P_{1,1}, P_{1,2}$ the prices of the first asset, by $P_{2,0}, P_{2,2}$ the prices of the second asset. Thus there are three repeat-sale observations, the first two are for the first asset and the last one is for the second asset.

$$Y = \begin{bmatrix} P_{1,1} / P_{1,0} \\ P_{1,2} / P_{1,1} \\ P_{2,2} / P_{2,0} \end{bmatrix}.$$

In this example, the estimators of equal-weighted portfolio returns in time period 1 and 2 are

$$\hat{g}_1^e = \frac{1}{2}\left(\frac{P_{1,1}}{P_{1,0}} + \frac{P_{2,2}}{P_{2,0}} \frac{1}{\hat{g}_2^e} \right), \quad \hat{g}_2^e = \frac{1}{2}\left(\frac{P_{1,2}}{P_{1,1}} + \frac{P_{2,2}}{P_{2,0}} \frac{1}{\hat{g}_1^e} \right). \tag{11}$$

The estimators of price-weighted portfolio returns are

$$\hat{g}_1^p = \frac{P_{1,1} + P_{2,2} / \hat{g}_2^p}{P_{1,0} + P_{2,0}}, \ \hat{g}_2^p = \frac{P_{1,2} + P_{2,2}}{P_{1,1} + P_{2,0} \hat{g}_1^p}. \tag{12}$$

Obviously the return estimators of both the equal-weighted and the price-weighted portfolio have natural interpretations. The return estimators of equal-weighted portfolio are averages of individual asset returns or proxies of them; the estimators of price-weighted portfolio equal the ratios of portfolio values or proxies of them. At the same time, calculating the estimators is easy because there are two equations for two return estimators of each portfolio.

II.4. A Finite Sample Problem and its Correction

Equation (6) shows that an observation with holding interval H_n directly appears in the estimator-defining equations for all time periods that belong to H_n. For example, an observation whose holding interval consisting of period 1 and 2 would be used to estimate $\hat{g}_1$ and $\hat{g}_2$, and therefore appears in the defining equations for both two periods. At the same time, from equation (6), the $\hat{g}_2$ appears in the defining equation of $\hat{g}_1$ and vice versa. Thus this observation is actually used twice in the estimation of $\hat{g}_1$: one time it directly appears in the defining equation of $\hat{g}_1$ and another time it is included in $\hat{g}_2$ and $\hat{g}_2$ appears in the defining equation of $\hat{g}_1$. A finite sample problem would rise when the portfolio consists of small amount of assets and the length of the repeat sale observation holding interval varies a lot. The observations that have long holding intervals may dominate those with short holding intervals when estimating the portfolio returns because they are used for much more times.

Using the small data set from last subsection as example, one can easily show that the compounded estimated return of the price-weighted portfolio for period 1 and 2 is $\hat{g}_1^p \hat{g}_2^p = (P_{1,2} + 2P_{2,2})/(P_{1,0} + 2P_{2,0})$. The actual compound return is already known from the data, which is $(P_{1,2} + P_{2,2})/(P_{1,0} + P_{2,0})$. Clearly the second asset are over-weighted in the GMM estimators, simply because it is less frequently traded and the corresponding repeat sale observation has longer holding interval.

Down-weighting repeat sale observations with longer holding intervals can solve the finite sample problem. Dividing each repeat sale observation with its length of holding interval, equation (6) changes to

$$\sum_{n \in O_t} w_n \frac{1}{T_n}\left(y_n / \prod_{s \in H_n} \hat{g}_s - 1 \right) = 0 \text{, for } t = 1,\ldots,T \text{ .} \tag{13}$$

Using equation (13) to estimate the same data set used earlier, the compounded estimated return of the two-asset price-weighted portfolio for two periods is $\hat{g}_1^p \hat{g}_2^p = (P_{1,2} + P_{2,2})/(P_{1,0} + P_{2,0})$, which exactly equals the actual one. After the correction, the equal-weighted GMM estimator is actually equivalent to the arithmetic-average equal-weighted estimator by Goetzmann and Peng (2000).

III. Simulation Test

III.1. Alternative Methods and Accuracy Measurements

The simulations test the performance of four alternative methods in the estimating of time series returns of the price-weighted index, i.e. the actual DJII. The first one is the finite sample version of the GMM method proposed in this paper (GMM). The second one is the version of the repeat sales regression (RSR) that can be justified as maximum likelihood estimators according to Goetzmann (1992). The third one is the instrumental variable version of the arithmetic repeat sales regression (ARS) proposed by Shiller (1991). The fourth one is a simple method that estimates a daily DJII return by averaging all available individual daily returns for that day (weighted by prices). Though the repeat sales regression essentially provides estimators of equal-weighted portfolio returns, I still use it as a benchmark because it is well known and widely used. The ARS is a natural benchmark since it provides estimators of price-weighted portfolio returns. The simple method would be a handy choice when the problem of data paucity is not serious, thus it is interesting to put the method in the simulation and test its usefulness.

It is important to make sure it is fair to put these four methods together in the simulations since each of them could have many variants. For example, the RSR has its variants like three-stage RSR and Bayes RSR; the ARS has its interval-weighted versions and hedonic versions. The GMM is also extendable to have similar variants. I consider it is fair to put these four methods together in a horse race first because they are all one-step methods while their variants typically involve more than one steps and extra regressions, and also because they provide estimators with obvious economic meanings while their variants generally don't.

Table 1 provides a simple comparison of the properties of these four methods. Among them, the GMM method, the ARS method, and the simple method provide estimators that are arithmetic means of individual asset returns, while the RSR provide geometric mean estimators. The GMM, the ARS, and the RSR estimate portfolio returns with regression, while the simple method doesn't use regression. All methods except the RSR use the natural prices instead of the logarithmic ones. The RSR is able to estimate returns for equal-weighted portfolios only and the ARS is able to value both equal-weighted and price-weighted portfolios, while the GMM and the simple methods are able to value equal-weighted, price-weighted, and other weighted portfolios. The RSR and the GMM methods both down-weight observations with longer holding intervals, while the ARS doesn't.[2]

[2] Shiller (1991) propose other variants that taking account of error heteroskedasticity for different observations but require extra regressions

There are four different measurements for the accuracy of a method. Specifically, the first measurement evaluates the overall accuracy of a method. It is the squared difference of the geometric mean of the estimated returns and that of the actual returns, which is calculated by

$$\left[\left(\prod_{t=1}^{T}\hat{g}_t\right)^{\frac{1}{T}}-\left(\prod_{t=1}^{T}g_t\right)^{\frac{1}{T}}\right]^2 .$$

The smaller is the squared difference, the more accurate is the method on valuing the portfolio's long term performance. The second measurement is the standard deviation of the estimated returns because a good method is expected to provide estimators whose standard deviation is closer to the actual one. The third measurement is the R^2 resulting from regression of actual return series upon estimated one, which captures the correlation between the actual and estimated returns. A good method is expected to have a higher R^2. The fourth measurement is the mean of squared errors (MSE) of estimated returns for all periods. It also helps to capture a method's period to period performance, and is calculated as

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{g}_t-g_t)^2 .$$

Among the four measurements, the first one, which evaluates a method's ability to measure a portfolio's overall performance, is considered the most important.

III.2. Simulation Procedure

The data are actual daily prices of 30 Dow Jones Industrial Index stocks from September to December 1999. There are 85 trading days (so 84 daily returns for DJII since the first day is

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and the estimators may no longer be arithmetic averages of individual asset returns.

the base period) and totally 2,550 daily prices for the 30 stocks. The basic approach of a simulation is to randomly select some prices from all 2,550 daily prices to construct an infrequent-transaction data set, then estimate the time series of DJII daily returns over the three months with different methods. Based on estimators, one is able to calculate the four different accuracy measurements for each method.

The accuracy of different methods may depend on the frequency of trading (the length of holding intervals) and maybe the percentage of liquid assets in the portfolio as well. Therefore the test procedure carefully controls the number of prices drawn from the actual data and the percentage of liquid assets in the portfolio. Specifically, there are two groups of simulations. In the first group, there is no liquid asset, and all "observed prices" are randomly drawn from actual prices. This group contains twelve scenarios in which the number of "observed prices" is 200, 400, 600, 800, 1000, 1200, 1400, 1600, 1800, 2000, 2200, and 2400 respectively, representing scenarios in which illiquid assets trade with different frequency. Smaller number of observed prices corresponds to scenario in which assets trade less frequently and vice versa. In the second group of simulations, firstly randomly select some stocks as "liquid assets", whose prices are observed over all sample periods. There are 4 scenarios in which the portfolio has different percentage of "liquid assets": 10%, 20%, 30%, and 40%. In each scenario, the numbers of prices drawn for other stocks, the "illiquid assets", are also controlled, being 10%, 20%, and 30% respectively. Consequently, there are total 12 different scenarios in the second group of simulations: percentage of "liquid assets" ranges over 10%, 20%, 30%, and 40%; and in each case, the number of observed prices for "illiquid assets", ranges over 10%, 20%, and 30%. The second group of simulations test performance of different methods in valuing portfolio consisting of both liquid and illiquid assets. For each scenario in the first and the second group, simulation is repeated for 100 times. Thus the reported statistics of accuracy measurement are averages over 100 simulations.

III.3. Simulation Results

Table 2 reports the simulation results for the four methods on valuing portfolios sorely consisting of "illiquid asset" portfolios. From the squared error of the geometric mean of return estimators, the GMM method persistently provides the most accurate measurement of portfolios' overall performance. For all twelve scenarios, the squared error of its geometric mean is always smaller than that of other methods. The running up is the ARS, followed by the RSR and then the simple method. The simple method perform poorly when assets trade not very frequently, and finally over-performs the RSR when the data missing is less than 7%. The standard deviations of estimators for all four methods generally decrease and converge to the actual standard deviation when more and more prices are observed. However, no method is obviously superior in the sense that having standard deviation much closer to the actual one.

Not a surprise, for all four methods the average R^2 increases with the number of "observed prices", and the average MSE decreases with it, which confirms that all methods better capture actual return's evolution when more prices are observed. At the same time, the GMM method persistently and obviously performs better than the ARS and slightly better than the RSR in terms of higher R^2. The simple method has high R^2 when assets trade very infrequently, which is actually misleading because there may be many periods for which the simple method is simply not able to provide estimators. The RSR is actually doing well when assets trade very infrequently. Its average MSE is smaller than that of GMM and ARS when the number of observed prices is less than 1200. As a conclusion, the GMM works well in capturing the evolution of daily returns for price-weighted portfolios consisting of illiquid assets only, at least when assets trade reasonably frequently.

Table 2 reports the simulation results on valuing portfolios consisting of both "liquid assets" and "illiquid assets". All methods seem more accurate than when estimating portfolios sorely consisting of illiquid assets, after the number of observed prices is controlled. For example, the average R^2 for the GMM to value portfolios containing 10% liquid assets and with 10% of illiquid assets' prices observed (corresponding to about 500 observed prices) is 49.85%, while the average R^2 for it to value portfolios sorely containing illiquid assets that have 600 observed prices is 37.46%. In conclusion, all methods are more accurate if some of assets in the portfolio trade very frequently, even though the total number of observed prices is low.

The GMM method is clearly superior to all other methods on measuring the overall performance of portfolios. Its squared error of geometric mean of returns is obviously smaller than that of other methods in all scenarios. At the same time, the GMM always has higher R^2 than other three methods, and it has smaller MSE than other methods except in several scenarios such as valuing portfolios consisting of 10% liquid assets and with 20% or 30% prices observed for illiquid assets.

In conclusion, the simulations confirm that the GMM method is clearly superior to other three methods: the ARS, the RSR, and the simple method, on measuring portfolios' overall performance and period to period evolution as well. It superior is more obvious when valuing price-weighted portfolios containing both liquid and illiquid assets. When valuing the portfolios sorely consisting of illiquid assets, the GMM method is obviously more accurate than the other methods at least when assets trade reasonably frequently.

IV. Possible Extensions

When asset characteristics are observable, the accuracy of portfolio return estimators may be improved. First, since asset characteristics are assumed to help to determine returns, knowing characteristics helps to obtain better proxies for each assets' single period returns, thus helps to obtain more accurate portfolio return estimators, say, hedonic GMM estimators. Second, knowing characteristics helps to differentiate assets from each other, which make it possible to correct the biased sample problem. The first subsection discusses the hedonic GMM estimators, and the second one discusses the correction of biased sample.

IV.1. Hedonic GMM estimators

Equation (7), (11), and (12) show that the estimator of g_t is an arithmetic average of individual single-period returns or proxies of them. If all proxies exactly equal actual individual single-period returns, the estimator would exactly equal the actual portfolio return. Clearly the accuracy of the portfolio return estimator depends on the accuracy of the proxies of individual single-period returns. When asset characteristics are not observable, assets are not differentiated from each other, in which case the best proxy of an asset's return is the expected portfolio return. However, when asset characteristics are observable, it is possible to get better proxies because the characteristics may systematically affect an asset's expected return.

The model assumes that expected deviation of an asset's return from the expected portfolio return is a function of the asset's characteristics. A three-stage procedure may be used to estimate the functional form and then provide more accurate estimators of the portfolio returns. In the first stage, estimating the model as if characteristics were not observable, which provides consistent estimators of the expected portfolio returns, denoted by $\hat{g}_t^1$ since this estimation is the first stage. The difference of an observed asset return (or compound return) from the estimated portfolio returns is.

$$y_n / \prod_{t \in H_n} \hat{g}_t^1 = y_n / \prod_{t \in H_n} \boldsymbol{g}_t \hat{e}_t = \prod_{t \in H_n} (\boldsymbol{h}_{n,t} \boldsymbol{e}_{n,t} \hat{e}_t)\,. \tag{14}$$

The term $\hat{e}_t$ is estimation error in the first stage.

For the purpose of simplicity, assume asset characteristics remain the same within each holding interval. Then for repeat-sale observation *n*, the deviations of its returns from the expected portfolio returns, i.e. $\boldsymbol{h}_{n,t}$ for $t \in T_n$, are constant over the whole holding interval. Then simplify the notation to $\boldsymbol{h}_n$. Clearly the term $\tilde{\boldsymbol{h}}_n \equiv \left(y_n / \prod_{s \in H_n} \hat{g}_s^1 \right)^{\frac{1}{T_n}}$ is a measurement of $\boldsymbol{h}_n$ with error contained.

Given the $\tilde{\boldsymbol{h}}_n$ and observable characteristics, the second stage estimates the functional form how the expected deviation of an asset's return depends on its characteristics.

$$\tilde{\boldsymbol{h}}_n = g(c_n)\boldsymbol{e}_n \text{ , with } E[\boldsymbol{e}_n | g(c_n)] = 1\,. \tag{15}$$

Here both parametric and non-parametric approaches may be used to estimate the functional form of $g(.)$. Denote by $\hat{\boldsymbol{h}}_n$ the estimator of $\boldsymbol{h}_n$, i.e. $\hat{\boldsymbol{h}}_n = \hat{g}(c_n)$.

The third stage defines $y'_n = y_n / \hat{\boldsymbol{h}}_n^{T_n - 1}$, and estimates the portfolio returns with y'_n instead of y_n.

$$\sum_{n\in O_t} w_{n,t}\left(y'_n /(\prod_{s\in H_n} \hat{g}^3_s) - 1 \right) = 0 \text{, for } t = 1,\ldots,T .$$

The estimator of portfolio return is

$$\hat{g}^3_t = \sum_{n\in O_t} w_{n,t}\left(\frac{y'_n}{\prod_{\{s/s\in H_n, s\neq t\}} \hat{g}^3_s} \right) = \sum_{n\in O_t} w_{n,t}\left(\frac{y_n}{\prod_{\{s/s\in H_n, s\neq t\}} \hat{g}^3_s \hbar_n} \right). \tag{16}$$

Clearly $\hbar_n \hat{g}^3_s$ is more accurate than $\hat{g}^1_s$ as the proxy for the asset's return in period s because it exploits the fact that asset characteristics help to determine an asset's return. Consequently the term $\frac{y_n}{\prod_{\{s/s\in H_n, s\neq t\}} \hat{g}^3_s \hbar_n}$ is a better proxy for the asset's return in time period t, and the $\hat{g}^3_t$, as average of individual single-period returns or their proxies, is therefore more accurate.

This procedure also provides estimators of expected returns for a sub-set of assets with specific characteristics. For example, in real estate research, this procedure is able to estimate housing index for not only a broad metropolitan but also a specific neighborhood within. Suppose an indicator variable equals to 1 if a house is in the neighborhood and 0 otherwise. The value of this indicator variable can be treat as a characteristic for a house. The three stage procedure proposed here is able to estimate the metropolitan index and the impact of the indicator variable on a house's return as well, which is the expected return deviation from the metropolitan index for houses in the neighborhood. Then the estimators of housing index for the neighborhood equal to the metropolitan index adjusted by the expected deviations from it for houses in the neighborhood.

IV.2. Correction for biased sample

The biased sample problem exists if the probability for transactions to take place is higher for one subset of assets in the portfolio than others. This problem may be corrected if the asset characteristics that help to differentiate the subset of assets from others are observable.

For example, suppose one is interested in a portfolio consisting of all houses in town A and town B. During the sample periods, a new company headquarters itself in town A but nothing similar happens in town B, which may causes much more house transactions in town A than town B. Therefore while most houses in town A are included in the repeat sales data, only some houses in town B are included because of much less transactions taking place there within the sample periods. Thus the transaction data are biased.

If house location is unobservable, houses in town A and town B can't be differentiated from each other. Then there may not be any way to correct for the biased sample. The estimated portfolio returns are actually for a portfolio that consists of more houses in town A than what is desired. However, if house location is observable, one can tell how many houses in each town are included in the repeat sales data. Suppose the actual numbers of houses in each town are roughly the same, but there are two times of houses in town A included in the repeat sales data than houses in town B. Based on the belief that houses in the same town follow the same return process, one can double the weights of the houses in town B that are included in the data during estimation, which makes the data provide equal numbers of samples for houses in each town. Therefore the estimated portfolio returns are for the portfolio that consists of same proportion of houses in both towns.

V. Conclusions

This paper proposes a new approach to value portfolios containing illiquid assets based on method of moment. The model of return process is meaningful and the GMM estimators have natural interpretations. All the estimators are arithmetic averages of individual asset returns (or their proxies) and strictly correspond to portfolio returns, which is an important improvement over the currently broadly used RSR method. This new approach provides estimators for returns of any arbitrary-weighted portfolio, including equal-weighted, price-weighted, and value-weighted portfolio, which few models claim to be able to do. This model accommodates the arithmetic-RSR proposed by Goetzmann and Peng (2000). Also, this model is flexible and very easy to extend. For example, it is able to estimate the portfolio returns with or without asset characteristic data, while the estimators could be more efficient if both price data and characteristic data are available. At the same time, the model may be able to provide more accurate estimators by correcting the sample bias problem that transactions may take place more likely on over-valued assets.

Simulations are used to test the accuracy of the GMM estimators proposed in this paper. The data are actual financial data: 2,550 daily prices of Dow Jones Industrial Index stocks over September 1999 to December 1999. The basic approach of a simulation is to randomly select some prices from all daily prices to construct an infrequent-transaction data set, then estimate the actual DJII daily returns with different methods. The accuracy of a method is measured with four statistics. They are the squared error of the geometric mean of return estimators, the standard deviation of the return estimators, the R^2 resulting from regression of the actual DJII daily returns upon the estimated time series returns, and the mean squared differences between the actual and the estimated returns. The simulations confirm that the GMM method is clearly superior to the RSR, the ARS, and the simple method that estimates a daily return by averaging all available individual daily returns for that day. The superiority of the GMM is more obvious on valuing price-weighted portfolios containing both liquid and illiquid assets.

Appendix: Estimation Algorithms

To estimate the returns of equal-weighted portfolios, define matrix X, Y, W and I as following. The X is a N by T dummy matrix. Its rows correspond to repeat-sale observations, and columns correspond to time periods. For row n, the first nonzero dummy appears in the position that corresponds to the time period $b_n + 1$, the time period immediately after the first sale of nth observation, and the last nonzero dummy appears in the position corresponding to s_n, the time period of the second sale. All elements between these two nonzero dummies also equal one, while other elements in this row are zero. As an example, if an asset was purchased at the end of time period 2 and sold at the end of time period 4, and $T=5$, its corresponding row in X is $(0,0,1,1,0)$. The Y is defined as a N by 1 vector whose nth element is y_n. The W is an N by N diagonal matrix whose nth element is $1/T_n$. The I is a N by 1 vector of 1.

Now, the estimator-defining equations for the equal-weighted portfolio can be written in matrix form as

$$X'WI = X'W \exp[\log(Y) - X \log(\hat{g})],$$

$$\text{or } X'W\{\exp[\log(Y) - X \log(\hat{g})] - I\} = 0 .$$

It is clear that there are T equations for T estimators. Though these equations are not linear, solving them with searching techniques may not be very difficult.

The price-weighted portfolio returns are even easier to estimate. Define a T by 1 vector $\boldsymbol{b}$ whose tth element is a reciprocal price index for time t,

$$\boldsymbol{b}_t \equiv 1/\prod_{s=}^{t} \hat{g}_s \ .$$

It is obvious that knowing $\boldsymbol{b}$ is equivalent to knowing $\hat{g}$. Define by Z a N by $T+1$ matrix whose rows correspond to repeat-sale observations, and columns correspond to time periods but start with time period 0. The b_n th element in row n equals $-B_n$, and s_n th element equals S_n, all other elements are 0. For example, for the data set used in earlier section to illustrate the estimators, the Z is

$$Z = \begin{bmatrix} -P_{1,0} & P_{1,1} & 0 \\ 0 & -P_{1,1} & P_{1,2} \\ -P_{2,0} & 0 & P_{2,2} \end{bmatrix}.$$

With matrix *X*, *Z*, and vector $\boldsymbol{b}$, the estimator-defining equations for price-weighted portfolio returns can be written in matrix form as

$$X'WZ\begin{bmatrix} 1 \\ \boldsymbol{b} \end{bmatrix} = 0\,.$$

They are linear equations and it is trivial to solve out $\boldsymbol{b}$. Once $\boldsymbol{b}$ is known, $\hat{g}$ is known. For example, $\hat{g}_1 = 1/\boldsymbol{b}_1$, $\hat{g}_t = \boldsymbol{b}_{t-1}/\boldsymbol{b}_t$ for $t > 1$.

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***Table 1.* Property Comparison of Alternative Methods**

This table compares the properties of four alternative methods. The first property is if the estimators are arithmetic averages of individual asset returns. The second one is if the method runs regression. The third one is if the method uses natural prices instead of logarithmic prices. The fourth to the sixth are if the method is able to value equal-weighted, price-weighted, or other-weighted portfolios respectively. The seventh property is if the method down weights the observations with long holding intervals. The "Y" represents "yes", and the "N" represents "no".

	Arithmetic average	Regression	Natural Price	Equal-weighted	Price-weighted	Other-weighted	Time Down weight
GMM	Y	Y	Y	Y	Y	Y	Y
ARS	Y	Y	Y	Y	Y	N	N
RSR	N	Y	N	Y	N	N	Y
Simple	Y	N	Y	Y	Y	Y	NA

***Table 2.* Simulation Results: Valuing Portfolios Sorely Consisting of Illiquid Assets**

This table reports simulation results for four estimation methods: the GMM, the ARS, the RSR, and the simple method. The basic procedure of a simulation is to randomly select N prices from 2550 daily prices for all 30 stocks in DJII over September to December 1999 to construct an infrequent-transaction data set, then estimate the time series of DJII daily returns with different methods. The simulation is run for 100 times for N equals to 200, 400, 600, 800, 1000, 1200, 1400, 1600, 1800, 2000, 2200, and 2400 respectively. All accuracy measurement numbers are averages for 100 simulations. The R^2 results from regression of actual daily returns upon estimated series of estimators. The MSE of return series is the average squared difference between actual and estimated returns.

N	200	400	600	800	1000	1200	1400	1600	1800	2000	2200	2400
Squared Error of the Geometric Mean of Estimated Return Series (in 0.00001%)												
GMM	11.3	1.99	1.70	1.05	0.61	0.44	0.35	0.26	0.18	0.13	0.10	0.05
ARS	19.4	3.51	2.36	1.49	1.24	0.98	1.02	0.85	0.67	0.51	0.45	0.23
RSR	86.6	10.59	4.31	2.68	1.38	0.89	0.78	0.68	0.65	0.62	0.59	0.55
Simple	194.8	92.75	51.09	24.61	12.71	9.25	4.92	3.60	2.24	1.33	0.69	0.28
Standard Deviation of Return Series (in 0.01%)												
Actual	1.06	1.06	1.06	1.06	1.06	1.06	1.06	1.06	1.06	1.06	1.06	1.06
GMM	5.81	2.42	1.73	1.44	1.29	1.21	1.17	1.13	1.11	1.09	1.07	1.06
ARS	7.00	1.96	1.51	1.31	1.20	1.15	1.11	1.09	1.07	1.06	1.05	1.04
RSR	4.97	3.00	2.00	1.60	1.37	1.26	1.20	1.15	1.12	1.10	1.08	1.07
Simple	1.96	2.19	2.01	1.76	1.50	1.33	1.23	1.16	1.12	1.08	1.06	1.04
R^2 (in percentage)												
GMM	4.10	18.42	37.36	53.58	65.87	74.72	81.76	86.69	90.92	94.01	96.67	98.76
ARS	2.60	11.58	27.58	43.40	57.47	68.70	76.99	83.25	88.31	92.09	95.44	98.16
RSR	1.63	19.88	37.79	52.82	64.35	73.06	79.71	84.78	89.15	92.18	94.75	96.81
Simple	20.37	19.90	25.89	34.57	47.15	58.68	69.03	76.75	83.49	88.30	92.32	95.87
MSE of Return Series (in 0.001%)												
GMM	336.0	50.08	19.40	9.94	5.70	3.70	2.47	1.69	1.11	0.70	0.45	0.17
ARS	484.9	81.08	29.19	14.44	8.00	4.96	3.30	2.21	1.46	0.94	0.63	0.25
RSR	253.1	31.31	14.63	8.29	5.22	3.57	2.53	1.81	1.26	0.81	0.69	0.42
Simple	31.5	39.90	32.29	21.38	12.10	7.50	4.68	3.15	2.08	1.38	0.87	0.47

***Table 3.* Simulation Results: Valuing Portfolios Consisting of Liquid and Illiquid Assets**

This table reports simulation results for four estimation methods: the GMM, the ARS, the RSR, and the simple method. A simulation consists of two steps. First, construct an infrequent-transaction data set by randomly selecting *x%* stocks as "liquid assets", which means all their prices are observable, and *y%* daily prices of the rest of stocks. Then estimate the actual DJII daily returns. The *x* equals 10, 20, 30, and 40, and *y* equals 10, 20, and 30 respectively. All accuracy measurement numbers are average for 100 simulations. The R^2 results from regression of actual daily returns upon estimated series of estimators. The MSE of return series is the average squared difference between actual and estimated returns.

(x, y)	(10, 10)	(10, 20)	(10, 30)	(20, 10)	(20, 20)	(20, 30)	(30, 10)	(30, 20)	(30, 30)	(40, 10)	(40, 20)	(40, 30)
Squared Error of the Geometric Mean of Estimated Return Series (in 0.00001%)												
GMM	2.34	0.97	0.81	1.32	1.04	0.60	0.96	0.59	0.51	0.83	0.44	0.38
ARS	5.13	2.29	1.74	3.28	2.42	1.69	2.46	2.01	2.18	3.88	2.02	2.24
RSR	5.57	2.49	1.20	4.03	1.75	1.34	1.96	1.74	1.32	1.82	1.26	0.81
Simple	13.02	9.18	6.16	5.84	4.46	4.31	3.06	2.71	2.91	2.20	2.18	1.62
Standard Deviation of Estimated Return Series (in 0.01%)												
GMM	1.48	1.38	1.29	1.30	1.25	1.21	1.22	1.18	1.17	1.16	1.15	1.14
ARS	1.93	1.60	1.38	1.55	1.37	1.28	1.37	1.27	1.22	1.26	1.21	1.18
RSR	1.45	1.33	1.23	1.28	1.21	1.17	1.19	1.15	1.13	1.13	1.12	1.11
Simple	1.53	1.48	1.41	1.30	1.28	1.24	1.21	1.17	1.17	1.14	1.13	1.13
R^2 (in percentage)												
GMM	49.75	57.18	67.75	65.44	70.03	76.80	74.87	77.70	82.53	82.01	84.10	87.00
ARS	31.09	43.76	58.51	47.42	57.20	68.58	59.72	67.45	75.11	69.22	75.65	80.92
RSR	46.38	54.47	65.13	62.43	67.18	73.46	71.93	75.22	80.08	79.74	82.03	84.88
Simple	42.14	46.58	54.42	60.82	63.13	67.14	71.20	72.72	76.26	79.43	80.33	81.93
MSE of Return Series (in 0.001%)												
GMM	11.19	8.31	5.40	5.92	4.70	3.38	3.75	3.09	2.36	2.42	2.08	1.67
ARS	26.00	14.41	7.88	12.75	8.05	5.11	7.53	5.27	3.69	4.90	3.54	2.64
RSR	11.53	8.15	5.34	6.18	4.84	3.64	4.03	3.29	2.54	2.62	2.24	1.85
Simple	13.84	11.93	9.14	6.73	6.02	5.07	4.23	3.78	3.26	2.69	2.53	2.32