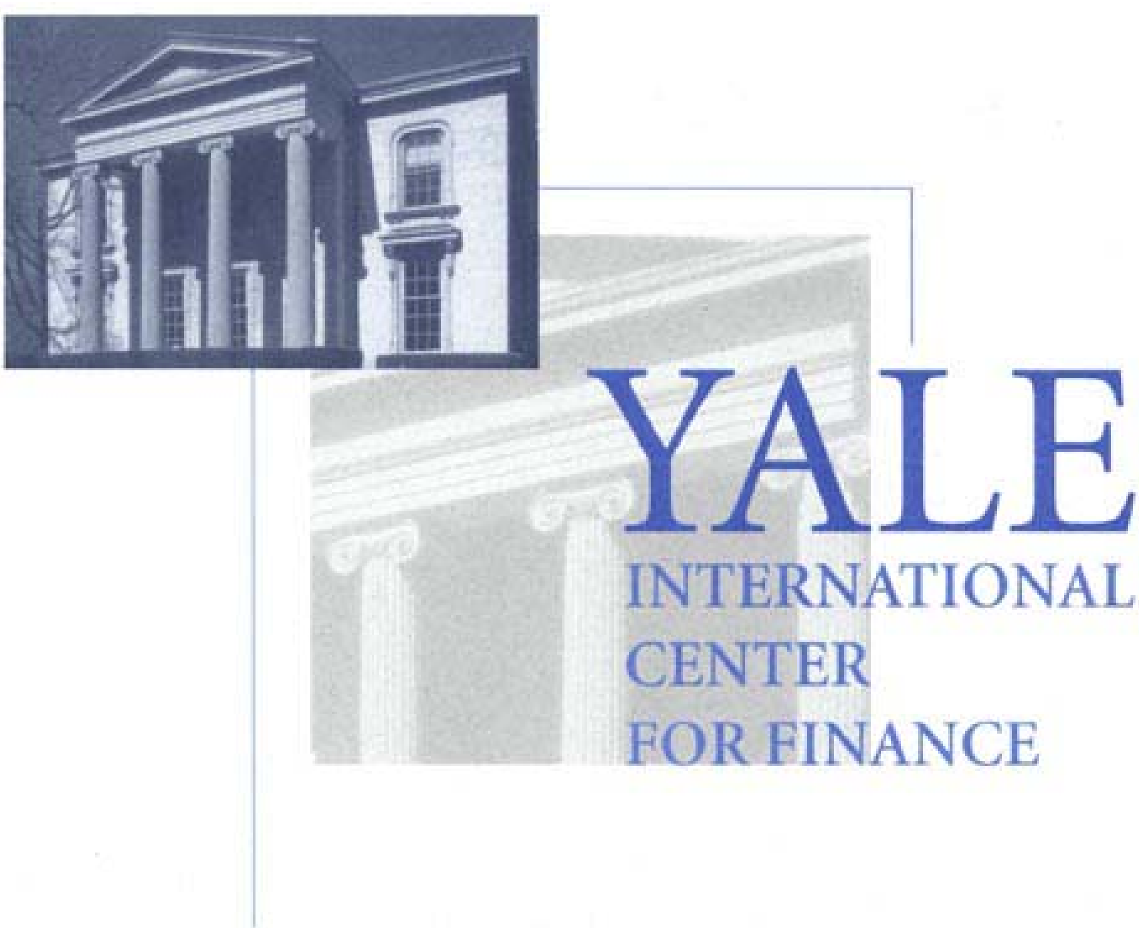

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A MODEL FOR PRICING STOCKS AND BONDS

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A Model for Pricing Stocks and Bonds

Abstract

This paper develops a tractable, dynamic, arbitrage-free model capable of jointly pricing a cross section of bonds and stocks. The bond pricing portion of the model produces the standard affine term-structure equations. It is then shown that a particular choice of dividend process, characterized by affine dividend yields, leads to stock prices that are exponential affine in the model's state variables. Importantly, the model allows for quite general interdependence between bond and stock prices. The paper also shows that an alternative modeling strategy for dividends, characterized by affine dividend growth rates, produces a less tractable model than the affine yield approach.

JEL Classification: G12, G13.

1 Introduction

This paper takes as its starting point the premise that it is desirable to have a joint bond-stock pricing model that has the following features: (i) empirical flexibility, (ii) theoretical coherence, and (iii) tractability. With this goal in mind, this paper proposes a parametric, no-arbitrage model for stock and bond prices. By developing a model which exhibits the above characteristics, this paper provides a platform on which future work in empirical asset pricing may be based. A desirable feature of doing empirical work within the framework of an asset pricing model is that a certain amount of economic structure is naturally imposed. The principle drawback of such an approach is that the asset pricing model in question may be difficult to work with, as well as overly restrictive.

However, as will become clear, the key advantages of the present model lie in its generality and its tractability. The general idea of the paper is to borrow the technology of continuous-time affine term-structure models, and to apply this technology to the pricing of stocks. In the present setting, stocks are differentiated from bonds in two ways:[1]

1. Bonds pay a terminal dividend of one dollar at some point in the future, whereas stocks pay a stochastic terminal dividend at some point (potentially infinitely far off) in the future.

2. In addition to their terminal dividend, stocks pay a continuous dividend stream, whereas bonds do not.

An extensive literature exists on pricing such bonds in an affine setting of the type employed in this paper. The key feature of such affine models is their tractability: Bond prices are exponential affine functions of the model's state variables, with coefficients that satisfy a set of ordinary differential equations. It turns out that with a prudent choice of the form of the dividend process, affine factor dynamics and an affine short rate also imply exponential affine stock prices, with coefficients that satisfy a set of algebraic equations. Indeed, this result is the distinguishing feature of the present approach: Since stock prices have the same general form as bond prices, the model is easy to work with, and all empirical techniques that have been developed for the estimation of term-structure models may be brought to bear in the present setting.

We start the modeling exercise by exogenously specifying a dividend process for stocks. A cross-section of stocks may be handled by endowing each stock with its own dividend process. The main result of the paper is to show that the exogenously specified dividend process for the stock actually takes on the following form

$$\text{Dividend} = \text{Affine Dividend Yield} \times \text{Stock Price}.$$

It is important to point out that the above relationship is a result rather than an assumption. We start with an exogenously given Dividend, and then solve for the stock price. Comparing

[1] One similarity between bonds and stocks is that neither is allowed to default. An extension of the present model that incorporates default risk is in Mamaysky (2002b).

the endogenously determined Stock Price to the exogenously specified Dividend, we then find that the above relationship holds. Going forward, let us refer to this as the *dividend yield* approach.

With our choice of dividend process, short rate process, and factor dynamics, it is shown that bond and stock prices are both exponential affine functions of the model's state variables. The above specification is done in such a way so as to insure that bond prices are exactly the same as they are in a standard affine term-structure model (see, for example, Duffie and Kan (1996)). Furthemore, as is typically done in the literature, the short-rate is assumed to be stationary, implying that bond prices can depend only on factors with stationary distributions. We will refer to such factors as the Y-type factors. On the other hand, stock prices may depend on the Y-type factors, as well as on factors that are non-stationary. We will refer to these non-stationary factors as the Z-type factors. It is this feature of the model that gives it substantial empirical flexibility: Stock prices (in all likelihood) contain random walk components, and we need a model that can accomodate this feature of the data. Since stock prices can depend both on the Y and Z-type factors, the pricing equations of the model make precise the interdependence that can exist between stock and bond prices in the economy.

To facilitate empirical implementation of the present model, we show that the total returns process for stocks (that is, a portfolio formed by initially holding one share of the stock and then reivesting all dividends back into the stock itself) also is exponential affine, once a straightforward change of variables has been performed on the model's Z-type factors. Since bond prices do not depend on these non-stationary factors, this change of variables leaves bond prices unaffected. Because data on total returns processes is readily available this transformation of the model allows for a natural empirical implementation. Also, it is shown that the total returns process for a portfolio of stocks is almost exponential affine (in a sense made precise in the paper). The one case in which the total returns process for a portfolio is also exponential affine is when all factors have constant volatilities.

This paper shows that the "essentially affine" pricing kernel proposed in Duffee (2001) and in Dai and Singleton (2001) can also be used in the present setting. As these papers show, in the term-structure context, this pricing kernel gives the model a great deal of empirical flexibility for fitting risk premia, a feature which should prove useful in a joint bond-stock pricing model. Along these lines, we should note that Bakshi and Chen (1997a,b) propose a pricing model similar to the one in this paper: Their model also has Y and Z-type factors, and is capable of pricing stocks and bonds.[2] However, they assume that the pricing kernel is given by the marginal rate of substitution of a representative CRRA investor with an exogenously specified consumption process. Their specification of the pricing kernel is unnecessarily restrictive and, along with certain other assumptions in their model, leads to the counterintuitive implication that no two stocks can have a different non-zero loading on the same Z-type risk factor. The present paper conveniently avoids such difficulties.

[2]See also the Brennan, Wang, and Xia (2001).

Furthermore, Bakshi and Chen (1997a,b) impose very strong restrictions on the dynamic behavior of the Y and Z-type factors in their model. In this paper, I derive a tractable pricing model which allows for far more general factor dynamics than in Bakshi and Chen (1997a,b). The increased flexibility of the present model's pricing kernel and factor dynamics will, in all likelihood, prove extremely useful once these types of models are confronted with the data.

A competing modeling approach to the one proposed in this paper takes the dividend growth rate, rather than the dividend yield, as the economic primitive. Indeed, in a discrete-time setting, Bekaert and Grenadier (2000) develop a pricing model with affine dividend growth rates. In this paper, I extend their results to a continuous-time setting, and am able to show that, in principle, there is an equivalence between the dividend yield approach advocated in this paper and the dividend growth rate approach of Bekaert and Grenadier.[3] This equivalence works as follows: An affine dividend yield model has a dividend growth rate counterpart, but with non-affine growth rates; likewise, an affine dividend growth rate model has a dividend yield counterpart, but with non-affine dividend yields. It turns out that the dividend yield approach naturally produces a model for stock prices, whereas the dividend growth rate approach naturally produces a model for price to dividend ratios (inverses of the dividend yields). Furthermore, affine dividend yields lead to stock prices that are exponential affine. However, in the case of affine dividend growth rates it is not clear whether or not price to dividend ratios have closed form solutions.[4] Of course, which of affine dividend yields and affine dividend growth rates provides a better fit to the data is an empirical question. However, it appears that a key advantage of the affine dividend yield approach advocated in this paper is the fact that it produces closed form stock prices.

The remainder of the paper proceeds as follows: Section 2 lays out the model. Section 3 shows how to compute bond and stock prices, and provides a discussion of the solution. Section 4 derives total returns processes from stock prices. Section 5 derives total returns processes for stock portfolios. Section 6 shows how to specify the pricing kernel, and how to move from the risk-neutral to the physical measure. Section 7 derives the dividend growth pricing model, and contrasts it to the dividend yield approach. Section 8 provides some examples of joint bond-stock pricing models. Section 9 concludes. The Appendix contains most of the proofs.

2 The Model

In this section, we will propose an economy which allows for the pricing of stocks and bonds in a unified and tractable setting. First, we discuss the primitives which drive uncertainty in the economy. Then we turn the pricing of financial securities. It will be seen that financial

[3]Strictly speaking this equivalence only holds when the dividend yield is forced to be positive, which in the present model does not have to be the case.

[4]In the discrete-time setting of Bekaert and Grenadier (2000), the price to dividend ratio is an infinite sum of exponential affine functions of the state variables.

securities have prices because they entitle their holders to future cashflows. It is the way in which these cashflows are specified that serves to differentiate bonds from stocks, and that allows for a tractable pricing model for the two asset classes. Finally, in this section we will establish some results about admissible parameter values for the model.

2.1 Economic Primitives

As a first step in the formulation of the model, we need to describe the way in which uncertainty evolves in the economy. In particular, we will assume that a set of factors exists, and that this set of factors drives all payoffs (as well as prices of risk). Hence all prices should be functions of these state variables. The first place where we make an "affine" assumption is in the specification of these factor dynamics.

To be more precise, we assume the existence of a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with an augmented filtration $\{\mathcal{F}_t : t \geq 0\}$. Furthemore, we assume the existence of a measure $\mathcal{Q}$, equivalent to $\mathcal{P}$, such that the discounted cumulative gains process for all securities (to be introduced shortly) in this economy are $\mathcal{Q}$ martingales. It is a well known results that, under some technical conditions, the existence of such a measure $\mathcal{Q}$ is equivalent to the absence of arbitrage opportunities (see Harrison and Pliska (1981) and Dybvig and Huang (1989)).

We assume the existence of an $N+M$ dimensional vector of state variables $X(t)$ which is admissible in the sense of Duffie and Kan (1996). By virtue of this assumption, $X(t)$ takes values in some open subset of $\mathbb{R}^{N+M}$. It will be convenient to write $X(t)$ as follows

$$X(t) = \begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix},$$

for an N dimensional vector $Y(t)$, and an M dimensional vector $Z(t)$. Letting $\tilde{W}(t)$ indicate a standard $N+M$ dimensional Brownian motion under $\mathcal{Q}$, the dynamics of $Y(t)$ and $Z(t)$ are given by

$$dY(t) = \tilde{K}_Y(\tilde{\Theta} - Y(t))dt + \Sigma_Y \sqrt{V(Y(t))} d\tilde{W}(t) \tag{1}$$

and

$$dZ(t) = \tilde{\mu} dt - \tilde{K}_Z Y(t) dt + \Sigma_Z \sqrt{V(Y(t))} d\tilde{W}(t) \tag{2}$$

respectively. Here $\tilde{K}_Y \in \mathbb{R}^{N\times N}$, $\tilde{\Theta} \in \mathbb{R}^N$, $\tilde{\mu} \in \mathbb{R}^M$, $\tilde{K}_Z \in \mathbb{R}^{M\times N}$, $\Sigma_Y \in \mathbb{R}^{N\times(N+M)}$, $\Sigma_Z \in \mathbb{R}^{M\times(N+M)}$, and $V(Y)$ is an $(N+M)\times(N+M)$ dimensional diagonal matrix, with elements along the diagonal given by

$$[V(Y)]_{nn} \equiv \alpha_n + \beta_n' Y$$

for $\alpha_n \in \mathbb{R}$ and $\beta_n \in \mathbb{R}^N$. Also let us define Σ_X as follows

$$\Sigma_X \equiv \begin{bmatrix} \Sigma_Y \\ \Sigma_Z \end{bmatrix}.$$

Hence Σ_X is an $(N+M)\times(N+M)$ matrix. Duffie and Kan (1996) discuss the restrictions which need to be placed on the parameters of the above processes to insure admissibility.

For future reference, let us define $\mathcal{D}_X$ as the Ito operator associated with X under $\mathcal{Q}$. We then will have that

$$\mathcal{D}_X\, f(Y,Z) = f'_Y \tilde{K}_Y(\tilde{\Theta} - Y) + f'_Z(\tilde{\mu} - \tilde{K}_Z Y)$$
$$+\frac{1}{2}\mathrm{tr}\Big(f_{YY'}\Sigma_Y V(Y)\Sigma'_Y\Big) + \frac{1}{2}\mathrm{tr}\Big(f_{ZZ'}\Sigma_Z V(Y)\Sigma'_Z\Big) + \mathrm{tr}\Big(f_{ZY'}\Sigma_Y V(Y)\Sigma'_Z\Big). \tag{3}$$

In the present economy, all securities are assumed to be default free. Furthemore, it is assumed that the model parameters are such that discounted gains processes associated with all securities in the economy are martingales under $\mathcal{Q}$.

2.2 Bonds

In the present setting, bonds are securities which entitle their owners to a dividend of \$1 at some point in the future. Since we have assumed that there is no default risk, bonds always pay the promised dividend. An alternative, riskless investment exists: money may be invested in a money market account, where it is assumed to grow at an instantaneous, riskless short rate of $r(t)$. This short rate is assumed to be an affine function of the model's Y-type factors.

In particular, we assume the existence of a short rate process $r(t)$ given by

$$r(t) \equiv r_0 + r'_Y Y(t), \tag{4}$$

where $r_0 \in \mathbb{R}$ and $r_Y \in \mathbb{R}^N$. A continuum of zero coupon bonds is assumed to exist with each bond paying its holder \$1 at some time T in the future. At time $t < T$, under the equivalent martingale measure $\mathcal{Q}$, the price of a bond maturing at time T is given by

$$P(t,T) = \mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r(u)du}\right]. \tag{5}$$

A general theory for the pricing of such bonds in the present setting is developed in Duffie and Kan (1996).

2.3 Stocks

Our approach to pricing stocks is exactly analogous to our approach for pricing bonds. We need to write down a dividend process (again, default free) which provides a reasonable model for the cashflows of actual equity-like securities, and then we need to determine the price of these cashflows. Of course, the specification of these cashflows can proceed in innumerable ways. However, of these, some ways of modeling equity dividends will naturally lead to more tractable models than others. Our goal, therefore, is to come up with a dividend specification which has enough flexibility to provide a reasonable fit for the data, but which also leads to a tractable pricing model.

With this goal in mind, we assume that the economy also contains $I > M$ classes of stocks, each class being characterized by its particular dividend process. Each stock in class

i is finitely lived, and is assumed to pay its holder a cumulative dividend through time t given by

$$\int_0^t D_i(u)du.$$

Furthermore, at some future time T, the stock is assumed to pay a terminal dividend given by $\bar{D}_i(T)$, after which time the stock will pay no more dividends. We assume that

$$D_i(t) \equiv (\delta_{0i} + \delta'_{Yi} Y(t)) \exp\Big(a(\delta_{0i}, \delta_{Yi}, C_i) \times t - B(\delta_{Yi}, C_i)' Y(t) - C'_i Z(t)\Big). \tag{6}$$

The values of $\delta_{0i} \in \mathbb{R}$, $\delta_{Yi} \in \mathbb{R}^N$, and $C_i \in \mathbb{R}^M$ are exogenously specified (subject to some restrictions which will be given later). For future reference, let us define $\delta_i(t)$ as

$$\delta_i(t) \equiv \delta_{0i} + \delta'_{Yi} Y(t). \tag{7}$$

For reasons which will be clear shortly, let us refer to $\delta_i(t)$ as the *instantaneous dividend yield* for stocks in class i.

Let us comment on the specification of the instantaneous dividend in (6) in light of the comments of the previous paragraph. On the one hand, the dividend process in (6) allows for $\{\delta_{0i}, \delta_{Yi}, C_i\}$ to be free parameters. Hence instantaneous dividends may load on stationary and on non-stationary factors in fairly general ways. On the other hand, to maintain tractability, we must restrict certain aspects of how the instantaneous dividend loads on the model's factors. These restrictions have two forms: First, $D_i(t)$ is assumed to be the product of an affine function with an expontial affine function; Second, certain aspects of how the exponential affine function loads on the economy's state variables are restricted (i.e. $a(\cdot)$ and $B(\cdot)$ are functions, given below, of the dividend parameters $\{\delta_{0i}, \delta_{Yi}, C_i\}$). As will become clear shortly, both of these restrictions are crucial for maintaining tractability of the pricing model. The hope is that these restrictions do not become overly burdensome when the model is confronted with the data. Also, in Section 7, we will discuss an alternative (and apparently less tractable) modeling strategy for dividends.

Returning to the specification of the dividend process, the functions $a(\cdot)$ and $B(\cdot)$ from equation (6) are assumed to satisfy the following

$$a_i = r_0 - \delta_{0i} + \tilde{\Theta}' \tilde{K}'_Y B_i + \tilde{\mu}' C_i - \frac{1}{2} \sum_{n=1}^{N+M} \Big([\Sigma'_Y B_i]_n + [\Sigma'_Z C_i]_n \Big)^2 \alpha_n, \tag{8}$$

$$0 = -r_Y + \delta_{Yi} + \tilde{K}'_Y B_i + \tilde{K}'_Z C_i + \frac{1}{2} \sum_{n=1}^{N+M} \Big([\Sigma'_Y B_i]_n + [\Sigma'_Z C_i]_n \Big)^2 \beta_n, \tag{9}$$

where we will write a_i and B_i as shorthand for $a(\delta_{0i}, \delta_{Yi}, C_i)$ and $B(\delta_{Yi}, C_i)$ respectively. Again, the importance of this assumption will become clear momentarily.

Furthemore, tractability of the pricing model forces us to assume that the terminal dividend $\bar{D}_i(T)$ is given by

$$\bar{D}_i(T) \equiv \exp\Big(a_i \times T - B'_i Y(T) - C'_i Z(T)\Big). \tag{10}$$

With this, we see that the factor dynamics in (1) and (2), as well as a choice of $\{\delta_{0i}, \delta_{Yi}, C_i\}$, fully determine the dividend process paid by a given stock. It should be emphasized that a_i and B_i are not choice variables, but instead must satisfy (8) and (9) above. Also note that equation (9) may not have a unique solution. This issue will be addressed in Section 3.1 of the paper.

Keep in mind that thus far we have done nothing more than write down a dividend process (albeit a carefully chosen one) for a given stock. We now need to compute the price of a stock with such a dividend stream. For $t < T$, let us write the price of a stock which expires at time T, with a dividend process given by $\{\delta_{0i}, \delta_{Yi}, C_i\}$, as $S_i(t,T)$. Notice that we must have $S_i(T,T) = \bar{D}_i(T)$. The cumulative gains process associated with this stock is given by

$$g_i^T(t) = \int_0^t e^{-\int_0^u r(s)ds} D_i(u)du + e^{-\int_0^t r(s)ds} S_i(t,T)$$

for $t < T$. Under the martingale measure $\mathcal{Q}$, discounted cumulative gains processes are martingales. Therefore we have that $g_i^T(t) = \mathbb{E}_t^{\mathcal{Q}}[g_i^T(T)]$. Simple algebra implies that this is equivalent to the following pricing equation for the stock:

$$S_i(t,T) = \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r(s)ds} D_i(u)du + e^{-\int_t^T r(s)ds}\bar{D}_i(T)\right]. \tag{11}$$

We now impose a *transversality* condition on the terminal dividend as follows

$$\lim_{T\to\infty} \mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r(s)ds}\bar{D}_i(T)\right] = 0. \tag{12}$$

We will show shortly some cases in which the transversality condition holds.

Given our dividend specification, it will later be shown that for all $T' \neq T$, $S_i(t,T) = S_i(t,T')$.[5] This observation allows us to refer to the price of any stock in class i as simply $S_i(t)$ regardless of that stock's maturity. Indeed, going forward we will simply refer to $S_i(t)$ as stock i, rather than as a stock from class i. Since (11) holds for all T, passing to the limit we find that

$$S_i(t) = \lim_{T\to\infty}\left(\mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r(s)ds} D_i(u)du\right] + \mathbb{E}_t^{\mathcal{Q}}\left[e^{-\int_t^T r(s)ds}\bar{D}_i(T)\right]\right).$$

Since $S_i(t)$ exists by assumption, the transversality condition in (12) allows us to conclude the the limit of the first expectation exists as well, and that it satisfies

$$S_i(t) = \lim_{T\to\infty} \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r(s)ds} D_i(u)du\right]. \tag{13}$$

The ability to express the stock price in the form of equation (13) is a direct consequence of the transversality condition imposed on the terminal dividend $\bar{D}_i(T)$. By requiring that

[5]As a preview of future results, note that this, with the boundary condition that $S_i(t,t) = \bar{D}_i(t)$, implies that $S_i(t,T) = \bar{D}_i(t)$ for all $t \leq T$.

the expected present value of this terminal dividend go to zero as its payment date becomes ever farther away, we in effect force the price of the stock to reflect only the present value of the future intertemporal dividend stream. In particular, the stock price is given by the limit of a sequence of stocks which only pay intermediate dividends.

With this, we are now ready to determine the price of an infinitely lived stock which pays a cumulative dividend through time t given by

$$\int_0^t D_i(u)du.$$

For now, let us refer to the time t price of this stock as $S_i^\infty(t)$. The discounted gains process for this stock is given by

$$g_i^\infty(t) = \int_0^t e^{-\int_0^u r(s)ds} D_i(u)du + e^{-\int_0^t r(s)ds} S_i^\infty(t).$$

Since we have that $g_i^\infty(t) = \mathbb{E}_t^Q[g_i^\infty(T)]$, the stock price must satisfy

$$S_i^\infty(t) = \mathbb{E}_t^Q\left[\int_t^T e^{-\int_t^u r(s)ds} D_i(u)du + e^{-\int_t^T r(s)ds} S_i^\infty(T)\right]. \tag{14}$$

Furthermore, we impose a transversality condition on this stock price as follows

$$\lim_{T\to\infty} \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds} S_i^\infty(T)\right] = 0. \tag{15}$$

Let us refer to any price process for the infinitely lived stock which satisfies condition (14) for all T and condition (15) as *admissible.* We are now faced with the question of whether an admissible price process is unique. The following proposition shows that this is indeed the case.

Proposition 1 *Let $S_i^\infty(t)$ be an admissible price process for an infinitely lived stock. Then $S_i^\infty(t)$ is unique, and furthermore $S_i^\infty(t) = S_i(t)$.*

Proof. Since $S_i^\infty(t)$ is admissible, it must satisfy (14) for all T. Taking the limit of (14) allows us to conclude that

$$S_i^\infty(t) = \lim_{T\to\infty}\left(\mathbb{E}_t^Q\left[\int_t^T e^{-\int_t^u r(s)ds} D_i(u)du\right] + \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds} S_i^\infty(T)\right]\right).$$

$S_i^\infty(t)$ exists by assumption, and the limit of the second expectation is equal to zero by assumption. Therefore, the limit of the first expectation must exist as well, and must satisfy the following:

$$S_i^\infty(t) = \lim_{T\to\infty} \mathbb{E}_t^Q\left[\int_t^T e^{-\int_t^u r(s)ds} D_i(u)du\right].$$

This implies that the admissible price process is unique. And from equation (13), we see that $S_i^\infty(t) = S_i(t)$. Q.E.D.

One role of the transversality condition for stocks in (15) is to guarantee that if an admissible price process exists, then it is unique. Also the dividend and stock transversality conditions (equations (12) and (15)) serve to guarantee that the price of an infinitely lived stock (if it exists) is equal to the limit of the prices of finitely lived stocks. As will be shown later, prices of the finitely lived stocks are easy to compute. Furthermore, it will be shown that the price of a finitely lived stock $S_i(t)$ is admissible as the price of an infinitely lived stock, thereby resolving the question of existence.

Note that our no-arbitrage arguments so far have left the limit outside of the expectation in (13). For a situation where $D_i(t) \geq 0$, the monotone convergence theorem allows us to move the limit inside the expectation. In cases where the instantaneous dividend may be negative, the question of whether or not this move can be done must be checked on a case by case basis. It should be noted, though, that the ability to express a stock price as

$$S_i(t) = \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^{\infty} e^{-\int_t^u r(s)ds} D_i(u) du \right]$$

is a result, rather than a primitive condition for no arbitrage. The primitive condition in the present economy, that discounted gains processes are martingales under $\mathcal{Q}$, only implies (13).

2.4 The Transversality Condition

At this point it is possible to give some cases of the model for which the transversality condition on the terminal dividend is satisfied.

Proposition 2 *If the $\delta_i(t)$ from equation (7) is such that $\delta_i(t) \geq \epsilon > 0$, then equation (12) holds.*

This will be the case if δ_{Yi} loads only on processes which are always positive, and if $\delta_{0i} > 0$. The following proposition gives another case in which the tranversality condition holds

Proposition 3 *Assume that $\tilde{K}_Y$ is diagonal, that Σ_Y has non-zero elements only along its upper left diagonal (i.e. $\{1,1\}$, $\{2,2\}$, etc.), that Σ_Z has non-zero elements only along its lower right diagonal (i.e. $\{M, N+M\}$, $\{M-1, N+M-1\}$, etc.), and that $V(Y)$ is the identity matrix. The transversality condition in (12) holds if and only if following parameter restriction is satisfied*

$$\delta_{0i} + \delta_{Yi}' \tilde{\Theta} + \sum_{n=1}^{N} \frac{[\delta_{Yi}]_n [\Sigma_Y]_{nn}^2}{[\tilde{K}_Y]_{nn}^2} \left(\frac{1}{2} [\delta_{Yi}]_n - [r_Y]_n + [C_i' \tilde{K}_Z]_n \right) > 0, \tag{16}$$

where $[\cdot]_n$ denotes the n^{th} element of a vector.

The parameter restrictions in this proposition imply that the joint distribution of $X(t)$ is Gaussian. In this case, the expectation in (12) can actually be computed in closed form (as is done in the Appendix), from which this result follows.

3 Bond and Stock Prices

Now that we have specified the payouts of all securities in the model, we are ready to compute bond and stock prices. This section presents general results about how bond and stock prices are obtained in the present setting. These results involve two components: First, prices are shown to satisfy a certain partial differential equation subject to some boundary conditions; Second, a certain integral, involving the bond or stock price, must be shown to be a martingale for the parameter values involved. More concrete examples of bond and stock prices in the model are given in Section 8. Also in Section 8, it is shown that the martingale conditions given in the propositions below are indeed satisfied for the example economies in question.

Bond prices in the model are identical to those of Duffie and Kan (1996), and so the following proposition is stated without proof (though the proof, given below, of the analogous proposition for stocks is very similar).

Proposition 4 *In the present economy, discounted gains processes for zero coupon bonds will be martingales if and only if the following conditions are satisfied:*

1. *Bond prices are given by*

$$P(t,T) = \exp\Big(A_T(t) - B_T(t)'Y(t)\Big), \tag{17}$$

where $A_T(t)$ and $B_T(t)$ solve the following ordinary differential equations

$$0 = -r_0 + \partial A_T(t)/\partial t - \tilde{\Theta}'\tilde{K}_Y' B_T(t) + \frac{1}{2}\sum_{n=1}^{N+M} [\Sigma_Y' B_T(t)]_n^2 \alpha_n, \tag{18}$$

$$0 = -r_Y - \partial B_T(t)/\partial t + \tilde{K}_Y' B_T(t) + \frac{1}{2}\sum_{n=1}^{N+M} [\Sigma_Y' B_T(t)]_n^2 \beta_n, \tag{19}$$

subject to the boundary conditions that $A_T(T) = 0$ and $B_T(T) = 0$.

2. *The following integral is a $\mathcal{Q}$ martingale:*

$$\int_0^t e^{-\int_0^u r(h)dh} \frac{\partial P_T}{\partial Y'} \Sigma_Y \sqrt{V(u)} d\tilde{W}(u). \tag{20}$$

Whether the integral in (20) is a martingale has to be verified on a case by case basis.

Let us consider how to price a stock in class i, which pays a terminal dividend at time T. This will also allow us to show that $S_i(t,T) = S_i(t,T')$ for all $T \neq T'$. We recall that the cumulative gains process for a stock in class i is given by

$$g_i^T(t) = \int_0^t e^{-\int_0^u r(s)ds} D_i(u)du + e^{-\int_0^t r(u)du} S_i(t,T)$$

where $S_i(T,T) = \bar{D}_i(T)$ by assumption. We then see that

$$\begin{aligned} dg_i^T(t) &= e^{-\int_0^t r(u)du}\Big(-r(t)S_i(t,T) + D_i(t) + (\mathcal{D}_X S_i + \partial S_i/\partial t)\Big)dt \\ &\quad + e^{-\int_0^t r(u)du}\frac{\partial S_i}{\partial X'}\Sigma_X\sqrt{V(t)}d\tilde{W}(t). \end{aligned} \tag{21}$$

A necessary condition for $g^T(t)$ to be a $\mathcal{Q}$ martingale is that it have a zero drift. Therefore, the stock price $S_i(t,T)$ must satisfy

$$\mathcal{D}_X S_i(t,T) + \partial S_i(t,T)/\partial t = r(t)S_i(t,T) - D_i(t), \tag{22}$$

subject to the boundary condition that $S_i(T,T) = \bar{D}_i(T)$. We recall that the requirement that the discounted gain process for a stock be a $\mathcal{Q}$ martingale is equivalent to a stock price of the form in (11). With this, we are ready to state the following proposition:

Proposition 5 *Given a dividend process*

$$\{\delta_{0i}, \delta_{Yi}, C_i\}$$

discounted gains processes for stocks in class i are $\mathcal{Q}$ martingales if and only if the following conditions are satisfied:

1. *The stock price $S_i(t,T)$ is given by*

$$S_i(t,T) = \exp\Big(a_i \times t - B_i'Y(t) - C_i'Z(t)\Big) \tag{23}$$

   where a_i and B_i satisfy equations (8) and (9) respectively.

2. *The following integral is a $\mathcal{Q}$ martingale:*

$$\int_0^t e^{-\int_0^u r(s)ds}\frac{\partial S_i}{\partial X'}\Sigma_X\sqrt{V(u)}d\tilde{W}(u). \tag{24}$$

Proof. We note that the stock price in (23) satisfies the partial differential equation in (22) as well as the boundary condition that $S_i(T,T) = \bar{D}_i(T)$. This, together with the fact that X lies in some open subset of $\mathbb{R}^{N+M}$, implies that the solution given in (23) is indeed the unique solution for the stock price. We note that

$$g_i^T(T') = g_i^T(t) + \int_t^{T'} dg_i^T(u)$$

where $dg_i^T(u)$ is given in equation (21). That $g_i^T(t) = \mathbb{E}_t^{\mathcal{Q}}[g_i^T(T')]$ implies (23) and (24) follows from the fact that an Ito integral is a martingale only if it has a drift of zero, and from the fact that (23) is the unique solution to the PDE in (22) which satisfies the appropriate boundary condition. The fact that (23) and (24) imply that $g_i^T(t)$ is a $\mathcal{Q}$ martingale is obvious. Q.E.D.

Note that the stock price at time t does not depend on the date of the terminal dividend. As has already been pointed out, this allows us to refer to the prices of all stocks in class i as simply $S_i(t)$. Thus far, we have only managed to price a stock in class i which pays a terminal dividend given by $\bar{D}_i(T)$. We also see from Proposition 5 that $\bar{D}_i(t) = S_i(t)$ for all t. Let us now assume that the parameters of the model are such that the transversality condition in (12) holds (see Propositions 2 and 3). It is easy to check that $S_i(t)$ is therefore an admissible price process for an infinitely lived stock (since, by construction, it satisfies equations (14) and (15)). By Proposition 1, we also see that $S_i(t)$ is the unique admissible price process for the infinitely lived stock. This leads us to the following result:

Corollary 1 *The price for an infinitely lived stock with dividend process* $\{\delta_{0i}, \delta_{Yi}, C_i\}$ *exists, is unique, and is given by* $S_i^{\infty}(t) = S_i(t)$.

3.1 Choosing a Solution for B_i in Equation (9)

With these results, we now turn to a discussion of how to select the appropriate solution to B_i in equation (9). With $C_i = 0$ and $\delta_{Yi} = 0$, a solution to the equation in (9) gives a particular solution to the ordinary differential equation for $B_T(t)$ in (19). However, subject to the boundary condition that $B_T(T) = 0$, (19) has a unique solution. Therefore, when the limit $\lim_{T\to\infty} B_T(t)$ exists, it will be unique, will have a time derivative equal to zero, and will satisfy equation (9) when C_i and δ_{Yi} are zero. If we require the risk loadings of a stock with $C_i = 0$ and with $\delta_{Yi} = 0$ to be the same as those of the limit of a sequence of bonds whose maturities go to infinity, we must select a solution to (9) which is equal to $\lim_{T\to\infty} B_T(t)$. Finally, to find a solution to (9) when C_i and δ_{Yi} are non-zero, we will select that solution whose limit as $C_i \to 0$ and as $\delta_{Yi} \to 0$ is equal to $\lim_{T\to\infty} B_T(t)$. The solution to (9) which has this property must be found on a case by case basis.

3.2 Discussion of the Dividend Specification

Thus far, we have seen that an admissible infinite horizon stock price satisfies the following

$$S_i(t) = \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^u r(s)ds} D_i(u)du + e^{-\int_t^T r(s)ds} S_i(T)\right].$$

Given the result in Proposition 5, we see that the instantaneous dividend process for the stock is actually equal to

$$D_i(t) = \delta_i(t)\, S_i(t). \tag{25}$$

This reveals that, going forward, we are able to refer to $\delta_i(t)$ as the instantaneous dividend yield on stock i. Indeed, we are free to choose this dividend yield as we wish via δ_{0i} and δ_{Yi}. Note that the relationship in (25) is very much a result, and not an assumption – that is, we specified the dividend $D_i(t)$ exogenously (in equation (6)), and then showed that this dividend specification just happens to satisfy the relationship in (25). Also note that, in addition to the choice of δ_{0i} and δ_{Yi}, we are able to choose the $M \times 1$ vector C_i as part of the exogenous specification of the instantaneous dividend process. This allows the instantaneous dividend, as well as the stock price, to depend on non-stationary factors, a feature of the model which is important for its ability to provide an adequate fit for stock price data.

Because a dividend of the form in equation (6) can be expressed as an exogenously specified dividend yield times the stock price, as in equation (25), this modeling method may be called the *dividend yield* approach. In Section 7, we will consider an alternative modeling approach for stock prices in which the dividend growth rate, rather than the dividend yield, will be specified exogenously.

3.3 Discussion of the Stock Price Integral

The result that a stock price $S_i(t)$ of the form in equation (23) is equal to the expectation in (14) may seem surprising at first. Indeed, a relationship of the form

$$\begin{aligned} \exp\Big(a \times t - b'Y(t) - c'Z(t)\Big) &= \\ &\mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T (\delta_0 + \delta_Y' Y(t)) \exp\Big(a \times u - b'Y(u) - c'Z(u)\Big) du\right] \\ &+\mathbb{E}_t^{\mathcal{Q}}\left[\exp\Big(a \times T - b'Y(T) - c'Z(T)\Big)\right] \end{aligned} \tag{26}$$

does not hold in general for the $Y(t)$ and $Z(t)$ processes with dynamics in (1) and (2). In fact, given a choice of $\{\delta_0, \delta_Y, c\}$, the above relationship holds if and only if the a and b coefficients in (26) satisfy equations (8) and (9). This is exactly what we have proven in Proposition 5. Of course, it is at this point obvious why a dividend process of the form $\{\delta_{0i}, \delta_{Yi}, C_i\}$, with a_i and B_i solving (8) and (9), was selected in the first place – this is the only dividend process which allows for a stock price which is exponential affine in the state variables.

To build further intuition about this result, it may be useful to show by direct computation that (for a special case of the model) the relationship in (26) does indeed hold.[6] For the purposes of this example, let us assume that the short rate $r(t)$ is always zero, and that $C_i = 0$. Furthermore, assume that there is a single factor $X(t)$ whose dynamics are given by

$$dX = \tilde{K}(\tilde{\Theta} - X(t))dt + \sigma d\tilde{W}(t).$$

Furthermore, we assume that $\delta_i(t) = X(t)$. From Proposition 5, we see that the price of such a stock (of any maturity) is given by

$$S_i(t) = \exp\Big(a_i \times t + X(t)/\tilde{K}\Big),$$

[6] I thank Monika Piazzesi for her help in formulating this example.

where $a_i = -\tilde{\Theta} - \frac{1}{2}\sigma^2/\tilde{K}^2$ and where $B_i = -1/\tilde{K}$. Note that the choice of a_i and the loading on $X(t)$ in the exponential *must* be chosen according to equations (8) and (9) in the paper. The reason for this restriction will become clear shortly.

With the specializations, we can rewrite (26) as

$$\exp\left(a_i \times t + X(t)/\tilde{K}\right) = \\ \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T X(u)\exp\Big(a_i \times u + X(u)/\tilde{K}\Big)du + \exp\Big(a_i \times T + X(T)/\tilde{K}\Big)\right]. \quad (27)$$

Let us now verify that this relationship does indeed hold by direct computation of the expectation on the right hand side of the equation. If the following condition holds

$$\int_t^T \exp(a_i \times u)\, \mathbb{E}_t^{\mathcal{Q}}\left[\Big|X(u)\Big| \exp\Big(X(u)/\tilde{K}\Big)\right] du < \infty \quad (28)$$

for all t, T, then Fubini's Theorem allows us to exchange the order of integration in (27). Hence the right hand side of (27) can be written as

$$\int_t^T \exp(a_i \times u)\, \mathbb{E}_t^{\mathcal{Q}}\left[X(u)\exp\Big(X(u)/\tilde{K}\Big)\right] du + \exp(a_i \times T)\, \mathbb{E}_t^{\mathcal{Q}}\left[\exp\Big(X(T)/\tilde{K}\Big)\right]. \quad (29)$$

We now note that conditional on $X(t)$, for $u > t$ we have that $X(u)$ is Normal with a mean given by

$$\mu_t(u) = e^{-\tilde{K}(u-t)}X(t) + \tilde{\Theta}\Big(1 - e^{-\tilde{K}(u-t)}\Big)$$

and a variance given by

$$v_t(u) = \frac{\sigma^2}{2\tilde{K}}\Big(1 - e^{-2\tilde{K}(u-t)}\Big).$$

Finally, we need to use the fact that for a Normal random variable with mean μ and variance v, and any constant c, the following relationships holds:

$$\begin{aligned} \mathbb{E}\left[Xe^{cX}\right] &= (\mu + cv)\, \exp\Big(c\mu + \frac{1}{2}c^2 v\Big), \\ \mathbb{E}\left[|X|e^{cX}\right] &= \mathbb{E}\Big[|\tilde{X}|\Big]\, \exp\Big(c\mu + \frac{1}{2}c^2 v\Big), \end{aligned}$$

where $\tilde{X}$ is Normally distributed with mean $\mu + cv$ and variance v. By virtue of this, we see that condition (28) holds, justifying the expression in (29). Using these, we can rewrite (29) as

$$\int_t^T \left(\mu_t(u) + \frac{v_t(u)}{\tilde{K}}\right) \exp\left(a_i \times u + \frac{\mu_t(u)}{\tilde{K}} + \frac{1}{2}\frac{v_t(u)}{\tilde{K}^2}\right) du \\ + \exp\left(a_i \times T + \frac{\mu_t(T)}{\tilde{K}} + \frac{1}{2}\frac{v_t(T)}{\tilde{K}^2}\right). \quad (30)$$

We next make the following observation:

$$\frac{d}{du}\exp\left(a_i \times u + \mu_t(u)/\tilde{K} + \frac{1}{2}v_t(u)/\tilde{K}^2\right) =$$
$$-\left(\mu_t(u) + \frac{v_t(u)}{\tilde{K}}\right)\exp\left(a_i \times u + \frac{\mu_t(u)}{\tilde{K}} + \frac{1}{2}\frac{v_t(u)}{\tilde{K}^2}\right). \tag{31}$$

Indeed, this is the crucial step in our example. The above holds if and only if $a_i = -\tilde{\Theta} - \frac{1}{2}\sigma^2/\tilde{K}^2$ and $B_i = -1/\tilde{K}$. This is the role that a dividend process of the form $\{\delta_{0i}, \delta_{Yi}, C_i\}$, with a_i and B_i set by equations (8) and (9), plays in the paper.

Using the relationship in (31), it is easy to check that the expression in (30) is actually equal to

$$\exp\left(a_i \times t + \frac{\mu_t(t)}{\tilde{K}} + \frac{1}{2}\frac{v_t(t)}{\tilde{K}^2}\right).$$

Observing that $\mu_t(t) = X(t)$ and that $v_t(t) = 0$, we see that the right hand side of (27) does indeed equal the left hand side.

Finally, we note that the transversality condition in (15) holds for the stock in question as long as $a_i < 0$. Since the example is a special case of the parameter restrictions in Propostion 3, we note the equivalence of this condition with the one given in equation (16). Taking limits in (30) we see that, indeed,

$$\begin{aligned} S_i(t) &= \lim_{T\to\infty} \mathbb{E}_t^Q\left[\int_t^T D_i(u)du\right] \\ &= \int_t^\infty \left(\mu_t(u) + \frac{v_t(u)}{\tilde{K}}\right)\exp\left(a_i \times u + \frac{\mu_t(u)}{\tilde{K}} + \frac{1}{2}\frac{v_t(u)}{\tilde{K}^2}\right)du. \end{aligned}$$

Note that the crucial step in this derivation is that the derivative condition in (31) holds. This condition will not hold in general, of course. However, it does hold for the no arbitrage stock price associated with a dividend process of the form $\{\delta_{0i}, \delta_{Yi}, C_i\}$.

4 Total Returns Processes

In practice, it is often inconvenient to work directly with the stock price and its associated dividend process. Indeed, it is often easier to find data on the total returns process associated with a given security than it is to obtain separate data on that security's price and dividend processes. The total returns process on a security is defined to be the value of a portfolio which initially holds a single share of the security in question, and then reinvests all dividends back into the security itself. In the present model, instead of working with stock prices and dividend streams directly, it is possible to work only with the associated total returns process. In this section, we show that it is possible to find a transformation of the original Z type state variables, under which total returns processes will have an exponential affine form very similar to that of stock prices.

Let us now consider the set I of infinitely lived stocks, each with a dividend process parameterized by $\{\delta_{0i}, \delta_{Yi}, C_i\}$. Define the total returns process on a stock i as the portfolio which at time 0 holds a single share of the stock, and then reinvests all future dividends back into stock i. Let $n_i(t)$ be the number of shares of stock i that would be held at time t, having followed this reinvestment strategy from time 0. Let us refer to the value of the total returns process as $s_i(t)$. Then its dynamics are given by

$$ds_i(t) = n_i(t)\Big(dS_i(t) + \delta_i(t)S_i(t)dt\Big) + \Big(s_i(t) - n_i(t)S_i(t)\Big)r(t)dt. \tag{32}$$

The full reinvestment condition implies that

$$n_i(t) = \frac{s_i(t)}{S_i(t)}.$$

Using this condition, and dividing both sides in (32) by $s_i(t)$, allows us to write that

$$\frac{ds_i(t)}{s_i(t)} = \frac{dS_i(t)}{S_i(t)} + \delta_i(t)dt. \tag{33}$$

With this, an application of Ito's lemma to $\log s_i(t)$ allows us to show that

$$s_i(T) = s_i(t)\exp\left(\int_t^T \left[\delta_i(u)du + \frac{dS_i(u)}{S_i(u)} - \frac{1}{2}\frac{d[S_i](u)}{S_i^2(u)}\right]\right), \tag{34}$$

where $[\cdot]$ refers to the quadratic variation process (see Protter (1995)). We now choose an arbitrary subset $\mathcal{M}$ of M stocks. Let us reference the stocks in $\mathcal{M}$ by $1, \ldots, M$. Furthermore let us define an $M \times M$ dimensional matrix C as follows

$$C \equiv \Big[C_1\, C_2\, \cdots\, C_M\Big]$$

and an $N \times M$ dimensional matrix d_Y as

$$\delta_Y \equiv \Big[\delta_{Y1}\, \delta_{Y2}\, \cdots\, \delta_{YM}\Big].$$

Finally, let us define an M dimensional set of state variables $z(t)$, such that $z(0) = Z(0)$, where the dynamics of the new set of state variables is given by

$$dz(t) = \tilde{\mu}dt - \tilde{k}_Z Y(t)dt + \Sigma_Z\sqrt{V(Y(t))}d\tilde{W}(t). \tag{35}$$

The $M \times N$ dimensional matrix $\tilde{k}_Z$ is defined as

$$\tilde{k}_Z \equiv \tilde{K}_Z + (C')^{-1}\,\delta'_Y. \tag{36}$$

We then have the following result:

Proposition 6 *For each stock i in $\mathcal{M}$, the total returns process in (34) is given by*

$$s_i(T) = s_i(t) \exp\left(a_i(T-t) + \int_t^T \delta_i(u)du - B_i'\Big(Y(T) - Y(t)\Big) - C_i'\Big(Z(T) - Z(t)\Big)\right) \tag{37}$$

Furthermore, the total returns process is equivalent to:

$$s_i(t) = \exp\Big((a_i + \delta_{0i}) \times t - B_i'Y(t) - C_i'z(t)\Big) \tag{38}$$

for a_i and B_i satisfying equations (8) and (9) respectively.

Note that the total returns process $s_i(t)$ looks just like a stock which pays no dividends, and whose coefficients satisfy equations (8) and (9), with δ_{0i} and δ_{Yi} set to zero, and with $\tilde{K}_Z$ replaced by $\tilde{k}_Z$.

This proposition allows us to estimate the model using data on only bonds and total returns of stocks. Total returns processes on stocks retain the exponential affine form, and hence allow us to use traditional techniques to estimate all of the relevant parameters of the model. The only exception to this is that only the $\tilde{k}_Z$ matrix will be observable from the data, and not the $\tilde{K}_Z$ and the δ_Y matrixes separately. Indeed estimating the model using only total returns processes, and empirically extracting the Y factors, provides a specification test of the model as follows: A projection of the actual dividend yields of securities used in model estimation on the extracted Y factors should produce a perfect fit. If not, then the model is misspecified. A test of this nature is conducted in Mamaysky (2002a).

Note that the transversality condition in Proposition 3 can be rewritten for all stocks i in $\mathcal{M}$ in terms of total returns processes as follows:

$$\delta_{0i} + \delta_{Yi}'\tilde{\Theta} + \sum_{n=1}^{N} \frac{[\delta_{Yi}]_n[\Sigma_Y]_{nn}^2}{[\tilde{K}_Y]_{nn}^2}\left(-\frac{1}{2}[\delta_{Yi}]_n - [r_Y]_n + [C_i'\tilde{k}_Z]_n\right) > 0 \tag{39}$$

using the relationship for $\tilde{k}_Z$ in equation (36) from above. Note that condition (36) is well-defined only if C is non-singular. Hence the transversality result in Proposition 3 holds even for singular C, whereas the condition in (39) holds only for an invertible C matrix.

5 Portfolios of Stocks

The previous section showed that a total returns process on a single stock has an exponential affine form, once a proper change of state variable is performed. In this section, we show that a similar representation for the total returns process for a portfolio of stocks is possible. The representation developed in this section is useful because empirical studies of stocks are often performed using stock portfolios, rather than individual stocks. Hence it is necessary to understand how the individual stocks with which we have been concerned until now aggregate into portfolios.

To proceed with this derivation, recall from (23) that the price of stock i is given by:

$$S_i(t,T) = \exp\Big(a_i \times t - B_i' Y(t) - C_i' Z(t)\Big).$$

Consider a portfolio which at time t holds $x_i(t)$ shares of stock i. Let us refer to the time t value of this portfolio as $\omega(t)$. Imposing the self-financing constraint, the value of this portfolio evolves according to

$$d\omega(t) = \sum_{i=1}^{I} x_i(t)\Big(dS_i(t) + S_i(t)\delta_i(t)dt\Big) + \Big(\omega(t) - \sum_{i=1}^{I} x_i S_i\Big) r(t)dt.$$

If the portfolio is fully invested in the I stocks, then the second term in the above equation drops out and we can write that

$$\frac{d\omega(t)}{\omega(t)} = \sum_{i=1}^{I} f_i(t)\left(\frac{dS_i(t)}{S_i(t)} + \delta_i(t)dt\right) \tag{40}$$

where f_i represents the fraction of the portfolio invested in the i^{th} stock, and is given by

$$f_i(t) \equiv \frac{x_i(t)S_i(t)}{\omega(t)}. \tag{41}$$

Note that the full investment condition implies that

$$\sum_{i=1}^{I} f_i(t) = 1$$

for all t. From equation (40) we see that the dynamics of a fully invested, self-financing portfolio are fully determined by the behavior of the $f(t)$ process (i.e. the vector obtained by stacking the $f_i(t)$'s).

With regard to a given weights process $f(t)$ let us define $\bar{\cdot}$ as the averaging operator. That is, for some quantity $q_i(t)$ associated with stock i, we define

$$\bar{q}(t) \equiv \sum_{i=1}^{I} f_i(t) q_i(t). \tag{42}$$

For example if we let b_i be given by

$$b_i \equiv \begin{bmatrix} B_i \\ C_i \end{bmatrix}$$

then $\bar{b}_i$ is given by

$$\bar{b}(t) = \sum_{i=1}^{I} f_i(t) b_i.$$

Note that $\bar{b}(t)$ has a time dependence, through the portfolio weights $f(t)$, even though each b_i is constant. With this we are now ready to state the main result of this section.

Proposition 7 *The time T value of a stock portfolio whose weights process is $f(t)$ is given by*

$$\begin{aligned}\log\frac{\omega(T)}{\omega(t)} &= \bar{\bar{a}}(t)\,(T-t)+\int_t^T\bar{\delta}(u)du-\bar{\bar{B}}(t)'\Big(Y(T)-Y(t)\Big)-\bar{C}(t)'\Big(Z(T)-Z(t)\Big)\\ &+\frac{1}{2}\int_t^T\sum_{n=1}^{N+M}[\Sigma_X]_n\left(\sum_{i=1}^{I}f_i(t)b_ib_i'-\bar{b}(t)\bar{b}(t)'\right)[\Sigma_X]_n'\beta_n'\left(\tilde{K}_Y\right)^{-1}\times\\ &\qquad\Big(\tilde{K}_Y\tilde{\Theta}du+\Sigma_Y\sqrt{V(u)}d\tilde{W}(u)\Big),\end{aligned}\tag{43}$$

where $[\cdot]_n$ indicates the n^{th} row of the matrix inside the brackets, and where $\bar{\bar{a}}(t)$ is given by

$$\bar{\bar{a}}(t)=r_0-\bar{\delta}_0+\tilde{\Theta}'\tilde{K}_Y'\bar{B}(t)+\tilde{\mu}'\bar{C}(t)-\frac{1}{2}\sum_{n=1}^{N+M}\Big[\bar{b}(t)'\Sigma_X\Big]_n^2\alpha_n,\tag{44}$$

and where $\bar{\bar{B}}(t)$ satisfies

$$\tilde{K}_Y'\bar{\bar{B}}(t)=r_Y-\bar{\delta}_Y(t)-\tilde{K}_Z'\bar{C}(t)-\frac{1}{2}\sum_{n=1}^{N+M}\Big[\bar{b}(t)'\Sigma_X\Big]_n^2\beta_n.\tag{45}$$

The proof is given in the Appendix.

By comparing the portfolio returns process in equation (43) to the total returns process for a single stock in equation (37) we see that they differ in two ways. First, for a given dividend process $\{\delta_{0i},\delta_{Yi},C_i\}$, the a_i and B_i functions for the stock total returns process satisfy (8) and (9) respectively. The analogous dividend process for a portfolio is given by $\{\bar{\delta}_0(t),\bar{\delta}_Y(t),\bar{C}(t)\}$. However, the portfolio's $\bar{\bar{a}}(t)$ and $\bar{\bar{B}}(t)$ functions satisfy equations (44) and (45) respectively. Furthermore, the portfolio total returns process has an additional factor which is not present in a total returns process for a single stock. This factor arises because, in general, $\bar{\bar{B}}(t)$ is not equal to $\bar{B}(t)$. As can be seen from equations (9) and (45), the reason for this is that $\bar{B}(t)$ contains a weighted sum of the form

$$\sum_{i=1}^{I}f_i(t)\Big[b_i'\Sigma_X\Big]_n^2$$

for all n, whereas $\bar{\bar{B}}(t)$ contains a sum of the form

$$\left[\left(\sum_{i=1}^{I}f_i(t)b_i\right)'\Sigma_X\right]^2.$$

Also, the portfolio's coefficients all have a time dependence due to the time variation in $f(t)$.

There is one case in which the total returns process on a stock portfolio is identical to the total returns of a single stock, whose dividend process is properly defined. The following corollary states this result, which follows directly from Proposition 7.

Corollary 1 *Assume that* $\beta_n = 0$ *for* $n = 1, \ldots, N + M$*. Also assume that the weights vector* $f(t)$ *is constant. Then the total returns process for a portfolio* $\omega(t)$ *is equal to the total returns process of a single stock with a dividend process given by* $\{\bar{\delta}_0, \bar{\delta}_Y, \bar{C}\}$*.*

The significance of this corollary is the following: If we are working with stock portfolios in a Gaussian setting, and if the portfolio weights remain relatively constant over time, then we may model the total returns processes for stock portfolios as if they were the total returns processes of single stocks. The interpretation of those single stocks' dividend processes is that their parameters represent the weighted average of the corresponsing dividend processes $\{\delta_{0i}, \delta_{Yi}, C_i\}$ for the stocks held in those portfolios. Hence, in such a setting, we are completely justified in modeling portfolios as if they were individual stocks. However, as Proposition 7 shows, this assumption is no longer valid if the conditions of the corollary are not, at least approximately, satisfied.

5.1 Examples of Simple Stock Portfolios

In this section, we show two examples of the implications of Proposition 7. First, we will construct an example of a Gaussian economy in which the corollary holds. Then, we will construct an example of an economy in which the corollary does not hold.

The Gaussian Case. Let us assume that $r(t) = 0$ and $\delta_i(t) = 0$ for all stocks. Also, the two state variables in the economy evolve according to

$$\begin{aligned} dY(t) &= \tilde{K}_Y(\tilde{\Theta} - Y(t))dt + \sigma d\tilde{W}(t), \\ dZ(t) &= -\tilde{K}_Z Y(t)dt. \end{aligned}$$

Let stock 1 have $C_1 \neq 0$. Then the stock's price is given by

$$S_1(t) = \exp\Big(a_1\, t - B_1 Y(t) - C_1 Z(T)\Big),$$

where the a_1 and B_1 coefficients satisfy

$$\begin{aligned} a_1 &= \tilde{\Theta}\tilde{K}_Y B_1 - \frac{1}{2}\sigma^2 B_1^2, \\ 0 &= \tilde{K}_Y B_1 + \tilde{K}_Z C_1. \end{aligned}$$

It is then easy to verify that

$$\frac{dS_1(t)}{S_1(t)} = -B_1 \sigma d\tilde{W}(t).$$

Let stock 2 have $C_2 = 0$. From the above, it is obvious that $a_2 = 0$ and $B_2 = 0$, and that $S_2(t) = 1$ for all t.

Let us assume we have a portfolio with a weight of f on stock 1 and a weight of $1 - f$ on stock 2. Then from equation (40), the dynamics of this portfolio are given by

$$\frac{d\omega(t)}{\omega(t)} = f\,\frac{dS_1(t)}{S_1(t)} = -f\, B_1 \sigma d\tilde{W}(t).$$

It is obvious that for stock 3 whose dividend process is characterized by $C_3 = f\,C_1$, we will have $B_3 = f\,B_1$. Then for stock 3 we will have

$$\frac{dS_3(t)}{S_3(t)} = -B_3\sigma d\tilde{W}(t) = -f\,B_1\sigma d\tilde{W}(t).$$

Hence the total returns process of a portfolio is exactly equal to the total returns process of a stock whose dividend coefficient is the weighted average of the dividend coefficients of the stocks (i.e. stock 1 and 2) in the portfolio.

The Non-Gaussian Case. We now assume that the two state variables in the economy evolve according to

$$\begin{aligned} dY(t) &= \tilde{K}_Y(\tilde{\Theta} - Y(t))dt + \sigma\sqrt{Y(t)}d\tilde{W}(t), \\ dZ(t) &= -\tilde{K}_Z Y(t)dt. \end{aligned}$$

For stock 1 with non-zero C_1, we have that

$$S_1(t) = \exp\Big(a_1\,t - B_1 Y(t) - C_1 Z(T)\Big),$$

where the coefficients satisfy

$$\begin{aligned} a_1 &= \tilde{\Theta}\tilde{K}_Y B_1, \\ 0 &= \tilde{K}_Y B_1 + \tilde{K}_Z C_1 + \frac{1}{2}\sigma^2 B_1^2. \end{aligned}$$

The dynamics of stock 1 are therefore given by

$$\frac{dS_1(t)}{S_1(t)} = -B_1\sigma\sqrt{Y(t)}d\tilde{W}(t).$$

For stock 2, if we set $C_2 = 0$, then we again find that $a_2 = 0$, $B_2 = 0$, and $S_2(t) = 1$ for all t. A portfolio of these stocks therefore evolves according to

$$\frac{d\omega(t)}{\omega(t)} = -f\,\frac{dS_1(t)}{S_1(t)} = -f\,B_1\sigma\sqrt{Y(t)}d\tilde{W}(t).$$

Say that stock 3 has $C_3 = f\,C_1$. Unlike the Gaussian case, we no longer have that $B_3 = f\,B_1$ (unless $f = 1$).

From the above dynamics for $\omega(t)$, we can solve for $\log\omega(t)$ to find that

$$d\log\omega(t) = -f\,B_1\sigma\sqrt{Y(t)}d\tilde{W}(t) - \frac{1}{2}f^2B_1^2\sigma^2Y(t)dt. \tag{46}$$

We now define $\bar{\tilde{a}}$ and $\bar{\tilde{B}}$ as in equations (44) and (45):

$$\begin{aligned} \bar{\tilde{a}} &= \tilde{\Theta}\tilde{K}_Y f B_1, \\ 0 &= \tilde{K}_Y\bar{\tilde{B}} + \tilde{K}_Z f C_1 + \frac{1}{2}\sigma^2 f^2 B_1^2. \end{aligned}$$

We can now write (46) as follows

$$
\begin{aligned}
d\log\omega(t) &= \bar{\tilde{a}}dt - \bar{\tilde{B}}dY(t) - fC_1 dZ(t) \\
&\quad -\Big(\bar{\tilde{a}}dt - \bar{\tilde{B}}dY(t) - fC_1 dZ(t) + fB_1\sigma\sqrt{Y(t)}d\tilde{W}(t) + \frac{1}{2}f^2B_1^2\sigma^2 Y(t)dt\Big).
\end{aligned}
$$

Some algebra then allows us to write

$$
d\log\omega(t) = \bar{\tilde{a}}dt - \bar{\tilde{B}}dY(t) - fC_1 dZ(t) + \Big(\bar{\tilde{B}} - fB_1\Big)\Big(\tilde{K}_Y\tilde{\Theta}dt + \sigma\sqrt{Y(t)}d\tilde{W}(t)\Big). \tag{47}
$$

Noticing that

$$
\bar{\tilde{B}} - fB_1 = \frac{1}{2}\frac{\sigma^2}{\tilde{K}_Y}\Big(fB_1^2 - f^2B_1^2\Big)
$$

we see that (47) satisfies Proposition 7.

6 Price of Risk

So far, the development of the model has proceeded under the risk-neutral measure – that is, we have assumed factor dynamics under the measure $\mathcal{Q}$, and derived stock and bond prices given the behavior of these factors. However, the ultimate usefulness of this model lies in its empirical implementation, for which we need to be able to work under the physical measure $\mathcal{P}$. Let us assume that the conditional expectation of $d\mathcal{Q}/d\mathcal{P}$ is given by

$$
\mathbb{E}_t^{\mathcal{P}}\left[\frac{d\mathcal{Q}}{d\mathcal{P}}\right] = \exp\left(-\frac{1}{2}\int_0^t \Lambda(X(u))'\Lambda(X(u))du - \int_0^t \Lambda(X(u))'dW(u)\right). \tag{48}
$$

Let us refer to $\Lambda : \mathbb{R}^{N+M} \to \mathbb{R}^{N+M}$ as the *price of risk* process. Then from the Girsanov Theorem (see Protter (1995)), we have that the $\mathcal{Q}$ Brownian motion $\tilde{W}$ has the following decomposition

$$
\tilde{W}(t) = W(t) + \int_0^t \Lambda(u)du, \tag{49}
$$

where $W(t)$ is a Brownian motion under $\mathcal{P}$.

Note that the exponential in (48) is, by definition, a $\mathcal{P}$ martingale. Indeed, this requirement places a restriction on our choice for the $\Lambda(X)$ function. We will impose the following additional restriction on $\Lambda(X)$: The drifts and volatilities of the Y and Z factors under the physical measure must be functions of the Y factors only. This restriction implies that the roles of the Y and Z factors do not change when we change measures. Let us refer to a price of risk process which satisfies both of these restrictions as *admissible.*

We will restrict our attention to admissible price of risk processes in the following class:

$$
\Lambda(Y) = V(Y)^{-\frac{1}{2}}\Big(\lambda_0 + \lambda_Y Y\Big), \tag{50}
$$

where $\lambda_0 \in \mathbb{R}^{N+M}$ and $\lambda_Y \in \mathbb{R}^{(N+M)\times N}$. We will see shortly that this choice satisfies the second condition for admissibility, though the martingale condition must be checked on a case by case basis. For a discussion of the restrictions that the martingale condition in (48) places on a price of risk process of the form in (50) see Duffee (2001) and Dai and Singleton (2001). For example, the price of risk process used in Dai and Singleton (2000) can be written in the following form

$$\Lambda(Y) = V(Y)^{\frac{1}{2}}\, l$$

for some $(N+M)$-dimensional vector l. The restrictions on (50) which give this price of risk process are

$$\lambda_0 = \begin{bmatrix} l_1\, \alpha_1 \\ \vdots \\ l_{N+M}\, \alpha_{N+M} \end{bmatrix} \qquad \lambda_Y = \begin{bmatrix} l_1\, \beta_1' \\ \vdots \\ l_{N+M}\, \beta_{N+M}' \end{bmatrix}.$$

This follows from the fact that $V(Y)$ is a diagonal matrix whose n^{th} element is $\alpha_n + \beta_n' Y$. For the case where V is the identity matrix, and therefore the factors are all jointly Gaussian, Dai and Singleton (2001) show that a price of risk process of the form in (50) satisfies the martingale condition in (48).

Given the factor dynamics under $\mathcal{Q}$ from (1) and (2), we have that under $\mathcal{P}$ the factor dynamics are given by

$$dY(t) = K_Y(\Theta - Y(t))dt + \Sigma_Y \sqrt{V(Y(t))}dW(t) \tag{51}$$

and

$$dZ(t) = \mu\, dt - K_Z Y(t)dt + \Sigma_Z \sqrt{V(Y(t))}dW(t). \tag{52}$$

The above matrixes (without the tilde to indicate that they are under the physical measure) are given by

$$\begin{aligned} K_Y &= \tilde{K}_Y - \Sigma_Y \lambda_Y, \\ \Theta &= K_Y^{-1}\left(\tilde{K}_Y \tilde{\Theta} + \Sigma_Y \lambda_0\right), \\ K_Z &= \tilde{K}_Z - \Sigma_Z \lambda_Y, \\ \mu &= \tilde{\mu} + \Sigma_Z \lambda_0. \end{aligned}$$

Given factor dynamics under both measures, it is straightforward to show that λ_0 and λ_Y satisfy the following:

$$\lambda_0 = \begin{bmatrix} \Sigma_Y \\ \Sigma_Z \end{bmatrix}^{-1} \begin{pmatrix} K_Y \Theta - \tilde{K}_Y \tilde{\Theta} \\ \mu - \tilde{\mu} \end{pmatrix}, \tag{53}$$

$$\lambda_Y = \begin{bmatrix} \Sigma_Y \\ \Sigma_Z \end{bmatrix}^{-1} \begin{pmatrix} \tilde{K}_Y - K_Y \\ \tilde{K}_Z - K_Z \end{pmatrix}. \tag{54}$$

Hence there exists a unique mapping between factor dynamics and the parameters of the price of risk process in (50).

6.1 The Pricing Kernel

Let us define the pricing kernel $m(t)$ as

$$m(t) \equiv \exp\left(-\int_0^t r(u)du\right) \mathbb{E}_t^{\mathcal{P}}\left[\frac{d\mathcal{Q}}{d\mathcal{P}}\right]. \tag{55}$$

It then follows from the requirement that discounted gains processes are martingales under $\mathcal{Q}$, than under $\mathcal{P}$ we have that stock prices satisfy

$$m(t)S_i(t) = \mathbb{E}_t^{\mathcal{P}}\left[\int_t^T m(u)\delta_i(u)S_i(u)du + m(T)S_i(T)\right], \tag{56}$$

and that bond prices satisfy

$$m(t)P(t,T) = \mathbb{E}_t^{\mathcal{P}}[m(t')P(t',T)] \tag{57}$$

for any $t, t' \geq t, T \geq t'$. From these it is obvious that under $\mathcal{P}$, $m(t)P(t,T)$ is a martingale, as is the following quantity for stocks

$$\int_0^t m(u)\delta_i(u)S_i(u)du + m(t)S_i(t). \tag{58}$$

From the definition of the pricing kernel in (55), we find that the dynamics of m are given by

$$\frac{dm(t)}{m(t)} = -r(t)dt - \Lambda(t)'dW(t). \tag{59}$$

Note that kernel innovations in general depend on all $N+M$ sources of uncertainty in the economy. With this, an application of Ito's lemma to (57) and (58) gives us the following proposition:

Proposition 8 *Assume that the price of risk process is given by (50). Given stock price dynamics of the form*

$$\frac{dS_i(t)}{S_i(t)} = \mu_{S_i}(t)dt + \sigma_{S_i}(t)'dW(t)$$

we have that

$$\mu_{S_i}(t) + \delta_i(t) - r(t) = \Lambda(t)'\sigma_{S_i}(t). \tag{60}$$

For bond price dynamics given by

$$\frac{dP(t,T)}{P(t,T)} = \mu_T(t)dt + \sigma_T(t)'dW(t)$$

we have that

$$\mu_T(t) - r(t) = \Lambda(t)'\sigma_T(t). \tag{61}$$

Note that this proposition justifies our use of the term *price of risk* for the Λ process.

Bond and stock prices are given in equations (17) and (23) respectively. An application of Ito's lemma to stock and bond prices shows that $\sigma'_{S_i} = -(B'_i\Sigma_Y + C'_i\Sigma_Z)\sqrt{V}$ and that $\sigma'_T = -B_T(t)'\Sigma_Y\sqrt{V}$. Using these in (60) and (61), as well as λ_0 and λ_Y from (53) and (54), we find that

$$\begin{aligned}\mu_{S_i}(t) + \delta_i(t) - r(t) &= -B'_i\Big((K_Y\Theta - \tilde{K}_Y\tilde{\Theta}) + (\tilde{K}_Y - K_Y)Y(t)\Big) \\ &\quad -C'_i\Big((\mu - \tilde{\mu}) + (\tilde{K}_Z - K_Z)Y(t)\Big), \qquad (62)\\ \mu_T(t) - r(t) &= -B_T(t)'\Big((K_Y\Theta - \tilde{K}_Y\tilde{\Theta}) + (\tilde{K}_Y - K_Y)Y(t)\Big). \qquad (63)\end{aligned}$$

Note that because of our price of risk specification, expected excess returns are driven exclusively by the Y factors, and are therefore stationary.

7 Modeling Dividend Growth Rates

The stock pricing model developed in this paper has taken as its starting point a dividend process of the form

$$D_i(t) \equiv \delta_i(t)S_i(t),$$

thereby taking the dividend yield $\delta_i(t)$ as an economic primitive. Going forward, let us refer to this as the *dividend yield* model to indicate that the dividend yield is the economic primitive in the modeling framework. However, an alternative derivation of a stock pricing model is possible. Instead of making the dividend yield a modeling primitive, we can make the dividend growth rate the modeling primitive. It is then possible to compute the dividend yield implied by a particular choice of dividend growth rate. In this section, I will develop a continuous time dividend growth model, and contrast this modeling approach to the one adopted in this paper.[7] In section 7.1, a model for the price to dividend ratio is developed, given an exogenous dividend growth process. In section 7.2, the dividend growth rate model is compared to the dividend yield model developed in the previous sections of this paper.

7.1 The Dividend Growth Rate as a Primitive

Let us assume that $D_i(t)$ is the instantaneous dividend on stock i, and that the dynamics of $D_i(t)$ under the equivalent martingale measure $\mathcal{Q}$ are given by

$$\frac{dD_i(t)}{D_i(t)} = \tilde{\mu}_{D_i}(X)\,dt + \sigma_{D_i}(X)'\,d\tilde{W}(t), \qquad (64)$$

where $X \in \mathbb{R}^{N+M}$ is the state vector, $\tilde{\mu}_{D_i} : \mathbb{R}^{N+M} \to \mathbb{R}$, $\sigma_{D_i} : \mathbb{R}^{N+M} \to \mathbb{R}^{N+M}$. It is assumed that for every i, the above dividend process is an admissible factor in the sense of Duffie and

[7]Note that this dividend growth rate model is the continuous time analog of the (discrete time) model in Bekaert and Grenadier (2000).

Kan (1996). Going forward, we will supress the i subscript. Given the dividend growth rate dynamics in (64), we see that the time t dividend is given by

$$D(t) = D(0)\exp\left\{\int_0^t \left(\tilde{\mu}_D - \frac{1}{2}\sigma_D'\sigma_D\right) du + \int_0^t \sigma_D'\, d\tilde{W}(u)\right\}. \tag{65}$$

We will refer to this type of model as the *dividend growth rate* model, to indicate that the economic primitive is the dividend growth rate. The first thing to note about the dividend growth rate model is that the instantaneous dividend $D(t)$ can never change sign. Indeed if we assume that $D(0) > 0$, then we see from (65) that $D(t) > 0$ for all t.

For an infinitely lived stock which pays an instantaneous dividend given by $D(t)$, the cumulative discounted gains process is given by

$$g(t) = \int_0^t e^{-\int_0^s r(u)du} D(s)ds + e^{-\int_0^t r(u)du} S(t)$$

where $S(t)$ is the stock price at time t. The requirement that under $\mathcal{Q}$ we should have $g(t) = \mathbb{E}_t^{\mathcal{Q}}[g(T)]$ is equivalent to the following condition on the stock price

$$S(t) = \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T e^{-\int_t^s r(u)du} D(s)ds + e^{-\int_t^T r(u)du} S(T)\right]$$

for all $t, T > t$. Let us define the price to dividend ration as

$$\pi(t) \equiv S(t)/D(t).$$

If we plug in the form of the dividend in (65) in to the above stock pricing equation, and rearrange terms, we find that under $\mathcal{Q}$ we must have that

$$\begin{aligned}\pi(t) \;\; = \;\; & \mathbb{E}_t^{\mathcal{Q}}\left[\int_t^T \exp\left\{-\int_t^s \left(r - \tilde{\mu}_D + \frac{1}{2}\sigma_D'\sigma_D\right) du + \int_t^s \sigma_D' d\tilde{W}(u)\right\} ds\right. \\ & \left. + \exp\left\{-\int_t^T \left(r - \tilde{\mu}_D + \frac{1}{2}\sigma_D'\sigma_D\right) du + \int_t^T \sigma_D' d\tilde{W}(u)\right\}\pi(T)\right]. \end{aligned} \tag{66}$$

Let us now impose a transversality condition on the price to dividend ratio as follows

$$\lim_{T\to\infty} \mathbb{E}_t^{\mathcal{Q}}\left[\exp\left\{-\int_t^T \left(r - \tilde{\mu}_D + \frac{1}{2}\sigma_D'\sigma_D\right) du + \int_t^T \sigma_D' d\tilde{W}(u)\right\}\pi(T)\right] = 0. \tag{67}$$

Note that (66) follows from the no-arbitrage condition and the dividend process in (65), whereas condition (67) is imposed to insure uniqueness of the price to dividend ratio, and hence plays a similar role to the stock price transversality condition in (15). Before we state the main result of this section, let us define a process $h(t)$ as follows

$$\begin{aligned} h(t) \;\; \equiv \;\; & \int_0^t \exp\left\{-\int_0^s \left(r - \tilde{\mu}_D + \frac{1}{2}\sigma_D'\sigma_D\right) du + \xi(s)\right\} ds \\ & + \exp\left\{-\int_0^t \left(r - \tilde{\mu}_D + \frac{1}{2}\sigma_D'\sigma_D\right) du + \xi(t)\right\}\pi(t), \end{aligned} \tag{68}$$

where $\xi(t)$ is given by

$$\xi(t) \equiv \int_0^t \sigma_D' \, d\tilde{W}(u). \tag{69}$$

We will need to assume that σ_D is such that $\xi(t)$ is an admissible factor in the sense of Duffie and Kan (1996). Therefore, it will be the case that $\xi(t)$ takes on values in some open subset of $\mathbb{R}$. The price to dividend ratio condition in (66) is equivalent to $h(t)$ being a $\mathcal{Q}$ martingale, i.e.

$$h(t) = \mathbb{E}_t^{\mathcal{Q}}[h(T)].$$

Note that for price to dividend ratios, $h(t)$ plays the role of the discounted cumulative gains process for stocks. Indeed, in the dividend growth rate model of (65), the condition that $g(t)$ is a $\mathcal{Q}$ martingale is equivalent to the condition that $h(t)$ is a $\mathcal{Q}$ martingale.

Let us refer to any price to dividend ratio $\pi(t)$ which satisfies (66) and (67) as *admissible.* We then have the following result:

Proposition 9 *Condition (66) holds if and only if*

1. *The price to dividend ratio $\pi(t)$ satisfies*

$$\mathcal{D}_\pi h(t) + \frac{\partial h(t)}{\partial t} = 0 \tag{70}$$

for the Ito operator $\mathcal{D}_\pi$ defined in the Appendix.

2. *The following integral is a $\mathcal{Q}$ martingale*

$$\int_0^t \left(\frac{\partial h}{\partial \xi} \sigma_D' + \frac{\partial h}{\partial X'} \Sigma_X \sqrt{V} \right) d\tilde{W}(u). \tag{71}$$

If there exists an admissible price to dividend ratio $\pi(t)$, then this is the unique admissible price to dividend ratio, and furthermore $\pi(t)$ is given by

$$\pi(t) = \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^\infty \exp \left\{ - \int_t^s \left(r - \tilde{\mu}_D + \frac{1}{2} \sigma_D' \sigma_D \right) du + \int_t^s \sigma_D' \, d\tilde{W}(u) \right\} ds \right]. \tag{72}$$

The proof of this proposition is in the Appendix.

Note that this proposition provides us with a method for finding the unique admissible price to dividend ratio (if one exists). First we note that a solution $\pi(t)$ to (70) (for all attainable X and ξ) such that the integral in (71) is a $\mathcal{Q}$ martingale, satisfies equation (66). Furthermore, if the above $\pi(t)$ satisfies the transversality condition in (67) then it is admissible. If so, then $\pi(t)$ is the unique admissible price to dividend ratio from the second part of Proposition 9.

Assuming that an admissible price to dividend ratio $\pi(t)$ does not depend on $\xi(t)$, we can write the partial differential equation in (70) as

$$0 = 1 - \left(r - \tilde{\mu}_D \right) \pi(t) + \frac{\partial \pi(t)}{\partial t} + \mathcal{D}_X \pi(t) + \sum_{n=1}^{N+M} \frac{\partial \pi(t)}{\partial X_n} \sigma_D' \sigma_{X_n} \tag{73}$$

for the $\mathcal{D}_X$ operator given in (3), and for

$$\sigma'_{X_n} = [\Sigma_X]_n \sqrt{V}.$$

According to the above proposition, any price to dividend ratio which satisfies (73), as well as the martingale condition in (71) and the transversality condition in (67), is the unique admissible price to dividend ratio.

Also, let us note that for an admissible price of risk process of the form in (50), we have that $d\tilde{W}(t) = dW(t) + \Lambda(t)dt$, where $W(t)$ is a standard Brownian motion under the physical measure $\mathcal{P}$. Therefore, the dividend process may be written as

$$\frac{dD(t)}{D(t)} = \tilde{\mu}_D dt + \sigma'_D \Big(dW(t) + \Lambda(t)dt \Big).$$

Hence we see that the dividend growth rate under the physical measure is given by

$$\mu_D = \tilde{\mu}_D + \sigma'_D \Lambda(t).$$

7.2 Comparison of Growth Rate and Yield Models

With the above proposition, we see that a well defined pricing model can be obtained by assuming an exogenous dividend growth rate process of equation (64). Instead of obtaining a stock price, this approach more naturally produces a stock price to dividend ratio $\pi(t)$. For the remainder of this discussion let us confine ourselves to the affine factor dynamics case, where the X vector follows the dynamics given in (1) and (2).

The choice between the two approaches therefore comes down to answering the following two questions:

1. Which approach provides a more natural economic model?

2. Which approach is the more tractable one?

Before discussing these issues, let us note that in the case where the dividend of the dividend yield approach may be positive or negative (as in the Gaussian case), the two modeling approaches are irreconcilably different since the dividend growth rate model can only accomodate dividends which do not change signs.

However, in the case where the dividend yield approach has a strictly positive dividend, it is the case that the two modeling approaches can be made to produce exactly the same answers. To see this, let us assume that factor dynamics for X are given by (1) and (2). Then we note that the dividend of the dividend yield approach is given by

$$D(t) = \delta(t) S(t)$$

with $\delta(t) \equiv \delta_0 + \delta'_Y Y(t)$ and $S(t)$ given by (23). If $\delta(t) > 0$, then an application of Ito's lemma allows us to express $D(t)$ in the form of equation (65) with the drift and volatility functions given by

$$\tilde{\mu}_D = r - \delta + \frac{1}{\delta} \left(\delta'_Y \tilde{K}_Y (\tilde{\Theta} - Y) - \mathrm{tr}\Big[\delta_Y B' \Sigma_Y V \Sigma'_Y \Big] - \mathrm{tr}\Big[\delta_Y C' \Sigma_Z V \Sigma'_Y \Big] \right), \qquad (74)$$

and

$$\sigma_D' = \frac{1}{S(t)} \frac{\partial S}{\partial X'} \Sigma_X \sqrt{V} + \frac{1}{\delta(t)} \delta_Y' \Sigma_Y \sqrt{V}. \tag{75}$$

Since by construction we have that

$$\pi(t) \equiv \frac{S(t)}{D(t)} = \frac{1}{\delta(t)},$$

it is easy to check that $1/\delta(t)$ does indeed satisfy the pricing equation in (73). Assuming that the model parameters are such that the other conditions of Proposition 9 are satisfied, we see that the two modeling approaches are indeed identical as long as the behavior of the dividend growth rate in (64) is appropriately chosen (i.e. is given by equations (74) and (75)).

In the case where we are restricted to strictly positive instantaneous dividends, the difference between the two modeling approaches comes from the way in which the dividend growth rate is modeled. When we are working in an affine setting, this choice comes down either

- to modeling the dividend yield as an affine process (i.e. the *dividend yield* model, where $\delta(t)$ is affine in X),
- or to modeling the dividend growth rate as an affine proces (i.e. the *dividend growth rate* model, where $\tilde{\mu}_D$ and ${\sigma_D}'\sigma_D$ are affine in X).

It is obvious from the above discussion that these two modeling alternatives are not self-consistent: either the dividend yield may be affine, or the dividend growth rate dynamics may be affine, but not both!

With this, we can now address the two questions brought up at the beginning of this section:

1. *More Natural Economic Model:* Given the restriction to affine factor dynamics for X, whether $\delta(t)$ is more naturally modeled as an affine process, or whether $\tilde{\mu}_D(t)$ and $\sigma_D(t)'\sigma_D(t)$ are more naturally modeled as affine processes is a question in need of an empirical answer.
2. *More Tractable Approach:* The benefit of modeling the dividend yield $\delta(t)$ as an affine process is that the resultant stock price is exponential affine. Indeed, this is a major benefit. It is easy to check that for an affine $\tilde{\mu}_D$ and ${\sigma_D}'\sigma_D$, the partial differential equation in (73) has neither an affine, nor an exponential affine solution. Whether or not the equation has a closed form solution at all is an open question.

Of course, which model is the appropriate one for a given application depends on the empirical behavior of dividend growth rates and of dividend yields. If both entities may be reasonably modeled as affine processes, then tractability dictates that the dividend yield model proposed in this paper is the better choice. However, if the affine model for the dividend yield provides a very poor empirical fit, then the dividend growth rate model should be used, despite the added computational burden. An empirical implementation of the dividend yield model proposed in this paper can be found in Mamaysky (2002a).

8 Examples

In this section, we give to examples of joint stock-bond pricing models using the framework set out in this paper. First, we will discuss a model with stacked Ornstein-Uhlenbeck type processes, and then one where the state variables follow independent Cox, Ingersoll, Ross type processes.

Keep in mind that to show that a given model is valid, the following conditions need to be checked:

C1. The factor dynamics in (1) and (2) must be admissible in the sense of Duffie and Kan (1996).

C2. The solution to the stock B_i coefficient equation in (9) must be chosen in accordance with the discussion in Section 3.1.

C3. The stock transversality condition in (15) must be satisfied.

C4. The bond integral in (20) must be a $\mathcal{Q}$ martingale.

C5. The stock integral in (24) must be a $\mathcal{Q}$ martingale.

C6. The price of risk process in (50) must be admissible, and in particular must satisfy the martingale condition in (48).

As general results regarding whether or not these conditions hold are not available, these conditions must be checked on a case by case basis. Some of these conditions are quite easy to verify, whereas others, such as the martingale conditions in (20) and (24), are more difficult. For the two cases presented below, all of these conditions are rather straightforward.

8.1 Ornstein-Uhlenbeck-Type Processes

This is the simplest case of the present model. Let us assume that factor dynamics satisfy the conditions in Proposition 3. With this, the Y and Z factors are jointly Gaussian, and trivially satisfy the Duffie and Kan (1996) conditions. The B_i solution to equation (9) is unique. Also the transversality condition for stocks holds as long as the parameter restrictions in Proposition 3 are satisfied. The martingale conditions in (20) and (24) hold once we realize that for an integral of the form

$$\int_0^t x(u)' d\tilde{W}(u) \tag{76}$$

to be a $\mathcal{Q}$ martingale, it is sufficient that

$$\mathbb{E}_0^{\mathcal{Q}}\left[\int_0^t x(u)'x(u)\,du\right] < \infty \tag{77}$$

for all t (see, for example, Lipster and Shirayev (2001)). Since the $x(u)$ integrands in (20) and (24) are lognormal, the above integrability condition is obviously satisfied. Finally, the price of risk process in (50) is admissible, as is shown in Dai and Singleton (2001).

With the technical conditions taken care of, it is easy to check that Proposition 4 implies that zero-coupon bond prices are given by $P(t,T) = \exp(A_T(t) - B_T(t)'Y(T))$, where

$$A_T(t) = -(r_0 + \tilde{\Theta}' r_Y)(T-t) + \tilde{\Theta}' B_T(t) + \frac{1}{2}\sum_{n=1}^{N} \sigma_{Yn}^2 \int_t^T [B_T(s)]_n^2 ds,$$

and where

$$[B_T(t)]_n = \frac{r_{Yn}}{\tilde{K}_{nn}}\left(1 - e^{-\tilde{K}_{nn}(T-t)}\right).$$

Also, from Proposition 5 and its corollary, we see that for a stock with a dividend process given by $\{\delta_{0i}, \delta_{Yi}, C_i\}$, which satisfies the conditions of Proposition 3, the stock price is given by $S_i(t) = \exp(a_i \times t - B_i' Y(t) - C_i' Z(t))$, where

$$a_i = r_0 - \delta_{0i} + \tilde{\Theta}' \tilde{K}_Y' B_i + \tilde{\mu}' C_i - \frac{1}{2}\sum_{n=1}^{N} \sigma_{Yn}^2 [B_i]_n^2 - \frac{1}{2}\sum_{m=1}^{M} \sigma_{Zm}^2 [C_i]_m^2,$$

and where

$$B_i = \left(\tilde{K}_Y'\right)^{-1}\left(r_Y - \delta_{Yi} - \tilde{K}_Z' C_i\right).$$

An empirical implementation of this type of model using U.S. data is available in Mamaysky (2002a).

8.2 CIR-Type Processes

Let us now assume that the n^{th} Y-type factor has the following dynamics

$$dY_n(t) = \tilde{K}_n\left(\tilde{\Theta}_n - Y_n(t)\right)dt + \sigma_{Yn}\sqrt{Y_n(t)}\,d\tilde{W}_n(t),$$

and that the m^{th} Z-type factor has the following dynamics

$$dZ_m(t) = \tilde{\mu}_m dt - [\tilde{K}_Z]_m Y(t) dt + \sigma_{Zm} d\tilde{W}_m(t).$$

Here $[\tilde{K}_Z]_m$ is a $1 \times N$ vector, which represents the m^{th} row of the $M \times N$ matrix $\tilde{K}_Z$. By assumption all processes have independent increments. Admissibility of the Y processes requires that $2\tilde{K}_{Yn}\tilde{\Theta}_n > \sigma_{Yn}^2$ for all n (see Cox, Ingersoll, Ross (1985) or Duffie and Kan

(1996)). This guarantees that $Y_n(t) > 0$ for all n, t.[8] For a stock with a dividend process $\{\delta_{0i}, \delta_{Yi}, C_i\}$, the solution for the B_i coefficients in (9) which is the limit of $B_T(t)$ is

$$[B_i]_n = \frac{1}{\sigma_{Yn}^2}\left(-\tilde{K}_{Yn} + \sqrt{\tilde{K}_{Yn}^2 + 2\sigma_{Yn}^2\Big(r_{Yn} - [\delta_{Yi}]_n - [\tilde{K}_Z' C_i]_n\Big)}\right). \tag{78}$$

Also we require that element by element, for all i,

$$r_Y - \delta_{Yi} - \tilde{K}_Z' C_i > 0,$$

which will insure that $B_i > 0$. We will also assume that the short-rate is strictly positive $r(t) > 0$. This implies that $r_0 \geq 0$ and that $r_Y \geq 0$, with at least one strict inequality. The transversality condition in (15) holds as long as the conditions of Proposition 2 are satisfied, which we will assume to be the case. To see that the integrals in (20) and (24) are martingales, we first must solve for the candidate bond and stock prices.

From Proposition 4, we see that bond prices in the present setting are given by $P(t,T) = \exp(A_T(t) - B_T(t)'Y(t))$ where

$$A_T(t) = -r_0(T-t) + \sum_{n=1}^{N} \frac{2\tilde{K}_{Yn}\tilde{\Theta}_n}{\sigma_{Yn}^2} \log\left(\frac{2\,c\,e^{\frac{1}{2}(T-t)(c+\tilde{K}_{Yn})}}{(c+\tilde{K}_{Yn})\,(e^{c(T-t)}-1) + 2\,c}\right).$$

and where

$$[B_T(t)]_n = \frac{2\left(e^{c(T-t)}-1\right) r_{Yn}}{(e^{c(T-t)}-1)\left(\tilde{K}_{Yn}+c\right) + 2c},$$

where

$$c = \sqrt{\tilde{K}_{Yn}^2 + 2\sigma_{Yn}^2 r_{Yn}}.$$

This implies that $A_T(t) < 0$ and $B_T(t) > 0$ for all t, T, and hence that $0 < P(t,T) \leq 1$ because $Y(t) > 0$. Thus the martingale condition in (20) holds by virtue of the sufficiency of condition (77), of the fact that $r(t) > 0$, and of the fact that $Y_n(t)$ has finite expectations for all t.

According to Proposition 5, a stock with a dividend process given by $\{\delta_{0i}, \delta_{Yi}, C_i\}$ has a price given by $S_i(t) = \exp(a_i \times t - B_i'Y(t) - C_i'Z(t))$, where

$$a_i = r_0 - \delta_{0i} + \tilde{\Theta}'\tilde{K}_Y' B_i + \tilde{\mu}' C_i - \frac{1}{2}\sum_{m=1}^{M} \sigma_{Zm}^2 [C_i]_m^2,$$

and where B_i is given by equation (78). Because $B_i > 0$ and because $Y(t) > 0$, we see that the stock price may be bounded as follows

$$S_i(t) < \exp\Big(a_i \times t - C_i' Z(t)\Big).$$

[8]For the price of risk process which will be used, this condition also insures that zero is inaccesible for the Y-type factors under both the physical and the risk-neutral measures.

From the dynamics of Z, we know that

$$Z(t) = Z(0) + \tilde{\mu}\, t - \tilde{K}_Z \int_0^t Y(s)ds + \Sigma_Z\Big(\tilde{W}(t) - \tilde{W}(0)\Big).$$

Let us also assume that

$$C_i' \tilde{K}_Z < 0$$

element by element. Since $Y(t) > 0$ by assumption, we will have that $C_i'\tilde{K}_Z \int_0^t Y(s)ds < 0$. From this we see that

$$0 < S_i(t) < \exp\Big(a_i \times t - C_i' \Big[Z(0) + \tilde{\mu}\, t + \Sigma_Z\Big(\tilde{W}(t) - \tilde{W}(0)\Big)\Big]\Big).$$

Since the $\tilde{W}(t)$ are standard Brownian motions, the variance of $S_i(t)$ is finite for all t, which together with the finiteness of the expectation of $Y_n(t)$ and the fact that $r(t) > 0$, implies that the integral in (24) is a $\mathcal{Q}$ martingale by vitrue of the condition in (77).

Finally, we assume that the price of risk process is given by $\Lambda(Y) = \sqrt{V(Y)}l$ for some $(N+M)$-dimensional vector l. This implies that

$$\lambda_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{N+1} \\ \vdots \\ l_{N+M} \end{bmatrix}, \quad \lambda_Y = \begin{bmatrix} l_1\, \beta_1' \\ \vdots \\ l_N\, \beta_N' \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{and} \quad \Lambda(Y) = \begin{bmatrix} l_1\sqrt{Y_1} \\ \vdots \\ l_N\sqrt{Y_2} \\ l_{N+1} \\ \vdots \\ l_{N+M} \end{bmatrix}.$$

In a setting with $l_m = 0$ for $m = N+1, \ldots, N+M$, it was shown in Cox, Ingersoll, Ross (1985) that this price of risk process is consistent with no-arbitrage, which implies that it satisfies the martingale condition in (48). However, if the right-hand side of (48) is a $\mathcal{P}$ martingale for $l_m = 0$ for $m = N+1, \ldots, N+M$, then clearly it is also a $\mathcal{P}$ martingale for $l_m \neq 0$. The discussion in Section 6 then tells us how to switch to the physical measure, as a function of l. In particular, we see that $\tilde{K}_Y\tilde{\Theta} = K_Y\Theta$, implying that the admissibility condition for Y-type processes applies under both measures.

9 Conclusion

This paper proposes a model for jointly pricing bonds and stocks. The model produces bond prices that are identical to those of affine term-structure models. Furthemore, by specifying an exogenous dividend process that is characterized by an affine dividend yield, the model produces stock prices that are exponential affine in the economy's state variables. This last feature of the model makes it quite tractable, and well suited for empirical implementation. Furthemore, the model is able to accomodate the pricing of a cross-section of bonds and stocks over time.

The modeling approach proposed in this paper leads to many interesting empirical questions. First of all, the present model may be estimated using data on government bonds and on stock portfolios. One empirical implementation (assuming constant volatility) of this model is in Mamaysky (2002a). However, many empirical implementations of the model are possible, and which is best suited for the data is an open question. Once the model is estimated, many questions of interest can then be addressed in a theoretically coherent manner. Examples of issues which may be addressed include: the dynamic behavior of expected returns, the degree of predictability which exists in bond and stock prices, the extent to which stocks and bonds share common factors, a measure of "duration" for stocks, commonality in stochastic volatility in bond and stock prices, cross-sectional dependence of stocks on the factors extracted from the model, as well as many others.

Also, many interesting theoretical questions remain open. An extension of the present model to allow for default risk is in Mamaysky (2002b). Also, the continuous-time dividend growth model proposed in this paper deserves further study. In particular, it would be interesting to characterize the growth rate processes which lead to closed form solutions for price to dividend ratios. Many technical questions remain about the behavior of affine processes of the type used in this paper. For example, checking whether the integrals in (20) and (24) are $\mathcal{Q}$ martingales for cases more general than the ones handled in this paper is non-trivial (see Duffie, Filipovic, and Schachermayer (2002) for example). A synthesis of the reduced form model proposed in this paper (modified to account for default risk, as in Mamaysky (2002b)) and the structural approach for the valuation of risky debt and equity pioneered by Merton (1974) seems a promising direction for future work. Also, it may be possible to generalize the dividend process proposed in this paper without sacrificing the model's tractability.

Appendix

A Preliminary Results

We will now state a useful result for future reference.

Lemma A.1 *Given a factor Y with dynamics*

$$dY(t) = K(\theta - Y(t))dt + \sigma dW(t),$$

where W is a vector of independent standard Brownian motions. We then have that

$$\int_t^T Y(s)ds \tag{79}$$

is Normally distributed with a mean of

$$\theta(T-t) + \frac{Y(t)-\theta}{K}\left[1 - e^{-K(T-t)}\right],$$

and with a mean zero component given by

$$\frac{\sigma}{K}\int_t^T \left[1 - e^{-K(T-u)}\right] dW(u),$$

whose variance is

$$\frac{\sigma\sigma'}{K^2}\left[(T-t) - \frac{2}{K}\left(1 - e^{-K(T-t)}\right) + \frac{1}{2K}\left(1 - e^{-2K(T-t)}\right)\right].$$

This result is standard, and therefore the proof is omitted.

B Proof of Proposition 2

Let us define $\zeta(t)$ as follows (note that the i subscript is suppressed)

$$\zeta(t) \equiv e^{\int_0^t \delta(u)du - \int_0^t r(u)du}\bar{D}(t).$$

It is straightforward to check that for the $\mathcal{D}_X$ operator of (3), the following equation holds:

$$\mathcal{D}_X\bar{D} + \partial\bar{D}/\partial t = \bar{D}(r-\delta).$$

Using this condition, an application of Ito's lemma reveals that

$$\zeta(T) = \zeta(t) + \mathcal{I}_t(T). \tag{80}$$

where

$$\mathcal{I}_t(T) \equiv \int_t^T e^{\int_0^h (\delta(u)-r(u))du}\frac{\partial\bar{D}(X,h)}{\partial X'}\Sigma_X\sqrt{V(u)}d\tilde{W}(u).$$

We know that $\mathcal{I}_t(T)$ is a local martingale (see Protter (1995)). However, $\zeta(t) \geq 0$ by construction. Hence $\mathcal{I}_t(T)$ is bounded below by $-\zeta(t)$, known at time t. Therefore, conditional on $\mathcal{F}_t$, $\mathcal{I}_t(T)$ is a supermartingale. Let us now show that $\delta(u) \geq \epsilon > 0$ is sufficient. From the definition of $\zeta(t)$ and (80) we can write that

$$e^{\int_0^T \delta(u)du - \int_0^T r(u)du} \bar{D}(T) = \zeta(t) + \mathcal{I}_t(T).$$

Then the left hand side above satisfies

$$e^{\int_0^T \delta(u)du - \int_0^T r(u)du} \bar{D}(T) > e^{\epsilon \times T - \int_0^T r(u)du} \bar{D}(T).$$

Hence we have that

$$e^{\epsilon \times T - \int_0^T r(u)du} \bar{D}(T) < \zeta(t) + \mathcal{I}_t(T).$$

Taking expectations we find that

$$e^{\epsilon \times T} \mathbb{E}_t^Q \left[e^{-\int_0^T r(u)du} \bar{D}(T) \right] < \zeta(t) + \mathbb{E}_t^Q [\mathcal{I}_t(T)] \leq \zeta(t) + \mathcal{I}_t(t),$$

where the last step follows from the fact that $\mathcal{I}_t(T)$ is a supermartingale. Hence we have that

$$0 < \mathbb{E}_t^Q \left[e^{-\int_0^T r(u)du} \bar{D}(T) \right] \leq e^{-\epsilon \times T} \Big(\zeta(t) + \mathcal{I}_t(t) \Big).$$

Since

$$\mathbb{E}_t^Q \left[e^{-\int_0^T r(u)du} \bar{D}(T) \right] = e^{-\int_0^t r(u)du}\, \mathbb{E}_t^Q \left[e^{-\int_t^T r(u)du} \bar{D}(T) \right],$$

the result obviously follows as $T \to \infty$.

Q.E.D.

C Proof of Proposition 3

We would like to show that

$$\lim_{T \to \infty} \mathbb{E}_0^Q \left[e^{-\int_0^T r(u)du} \bar{D}_i(T) \right] = 0$$

if and only if condition (16) holds. First note that

$$\int_0^T r(u)du = r_0 + \sum_{n=1}^N r_{Yn} \int_0^T Y_n(u)du.$$

Using the results of Lemma A.1

$$\int_0^T Y_n(u)du = \tilde{\Theta}_n T + \frac{Y_n(0) - \tilde{\Theta}_n}{[\tilde{K}_Y]_{nn}} \left(1 - e^{-[\tilde{K}_Y]_{nn} T}\right) + \frac{\sigma_{Yn}}{[\tilde{K}_Y]_{nn}} \int_0^T \left(1 - e^{-[\tilde{K}_Y]_{nn}(T-u)}\right) d\tilde{W}_n(u).$$

keeping in mind that we are working under the equivalent martingale measure Q. The log terminal dividend is given by

$$\log \bar{D}_i(t) = a_i \times t - B_i' Y(t) - C_i' Z(t),$$

where a_i and B_i satisfy (8) and (9) respectively. Also we have that

$$\begin{aligned} Y_n(t) &= e^{-[\tilde{K}_Y]_{nn}t}\left(Y_n(0)+\tilde{\Theta}_n\left(e^{[\tilde{K}_Y]_{nn}t}-1\right)\right)+\sigma_n e^{-[\tilde{K}_Y]_{nn}t}\int_0^t e^{[\tilde{K}_Y]_{nn}u}d\tilde{W}_n(u), \\ Z_m(t) &= Z_m(0)+\tilde{\mu}_m t-\sum_{n=1}^N [\tilde{K}_Z]_{mn}\int_0^t Y_n(u)du+\sigma_{Zm}(\tilde{W}_m(t)-\tilde{W}_m(0)), \end{aligned}$$

where $[\cdot]_{mn}$ indicates the element in the m^{th} row and n^{th} column of the matrix. Using these, some algebra reveals that for $\exp(\phi(t)) \equiv \exp(-\int_0^t r(u)du)\bar{D}_i(t)$, we have that

$$\begin{aligned} \phi(t) = &\left[a_i - r_0 - C_i'\tilde{\mu} - \sum_{n=1}^N (r_{Yn} - [C_i'\tilde{K}_Z]_n)\tilde{\Theta}_n\right] t \\ &- \sum_{n=1}^N [B_i]_n \left(e^{-[\tilde{K}_Y]_{nn}t}Y_n(0) + \tilde{\Theta}_n(1-e^{-[\tilde{K}_Y]_{nn}t})\right) - C_i'Z(0) \\ &- \sum_{n=1}^N (r_{Yn} - [C_i'\tilde{K}_Z]_n)\frac{Y_n(0)-\tilde{\Theta}_n}{[\tilde{K}_Y]_{nn}}(1-e^{-[\tilde{K}_Y]_{nn}t}) \\ &- \sum_{n=1}^N \left([B_i]_n\sigma_{Yn} - (r_{Yn} - [C_i'\tilde{K}_Z]_n)\frac{\sigma_{Yn}}{[\tilde{K}_Y]_{nn}}\right)\int_0^t e^{-[\tilde{K}_Y]_{nn}(t-u)}d\tilde{W}_n(u) \\ &- \sum_{n=1}^N (r_{Yn} - [C_i'\tilde{K}_Z]_n)\frac{\sigma_{Yn}}{[\tilde{K}_Y]_{nn}}\int_0^t d\tilde{W}_n(u) - \sum_{m=1}^M [C_i]_m\sigma_{Zm}\left(\tilde{W}_m(t)-\tilde{W}_m(0)\right). \end{aligned}$$

Also note the following

$$\begin{aligned} \mathbb{E}_0^Q\left[\left(\int_0^t e^{-[\tilde{K}_Y]_{nn}(t-u)}d\tilde{W}_n(u)\right)^2\right] &= \int_0^t e^{-2[\tilde{K}_Y]_{nn}(t-u)}du \\ &= \frac{1}{2[\tilde{K}_Y]_{nn}}\left(1-e^{-2[\tilde{K}_Y]_{nn}t}\right), \\ \mathbb{E}_0^Q\left[\int_0^t e^{-[\tilde{K}_Y]_{nn}(t-u)}d\tilde{W}_n(u) \times \int_0^t d\tilde{W}_n(u)\right] &= \int_0^t e^{-[\tilde{K}_Y]_{nn}(t-u)}du \\ &= \frac{1}{[\tilde{K}_Y]_{nn}}\left(1-e^{-[\tilde{K}_Y]_{nn}t}\right). \end{aligned}$$

Recall that by assumption $\tilde{W}_n$ is independent of $\tilde{W}_m$. Also because $\phi(t)$ is Normally distributed, we have that

$$\mathbb{E}_0^Q[\exp(\phi(t))] = \exp\Big\{\mathbb{E}_0^Q[\phi(t)] + \frac{1}{2}\mathrm{Var}_0^Q(\phi(t))\Big\}.$$

We now take the expectation of $\xi(t)$, and drop all terms which are $o(t)$ since these become negligible as we pass to the limit. We then find that for large T we have that

$$\mathbb{E}_0^Q\left[\exp(\phi(T))\right] = \exp\left\{\left(a_i - r_0 - C_i'\tilde{\mu} - \sum_{n=1}^{N}(r_{Yn} - [C_i'\tilde{K}_Z]_n)\tilde{\Theta}_n\right) T \right.$$
$$\left. + \frac{1}{2}\sum_{n=1}^{N}\left[(r_{Yn} - [C_i'\tilde{K}_Z]_n)\frac{\sigma_{Yn}}{[\tilde{K}_Y]_{nn}}\right]^2 T + \frac{1}{2}\sum_{m=1}^{M} C_m^2 \sigma_{Zm}^2 T\right\}. \tag{81}$$

From equations (8) and (9) we have that

$$\begin{aligned} a_i &= r_0 - \delta_{0i} + \tilde{\Theta}'\tilde{K}_Y' B_i + \tilde{\mu}' C_i - \frac{1}{2}\sum_{n=1}^{N}\sigma_{Yn}^2 [B_i]_n^2 - \frac{1}{2}\sum_{m=1}^{M}\sigma_{Zm}^2 [C_i]_m^2, \\ B_i &= (\tilde{K}_Y')^{-1}(r_Y - \delta_{Yi} - \tilde{K}_Z' C_i). \end{aligned}$$

Using these in (81), we find that for large T

$$\mathbb{E}_0^Q[e^{\phi(T)}] = \exp\Big\{-\left(\delta_{0i} + \delta_{Yi}'\tilde{\Theta} + \sum_{n=1}^{N}\frac{\sigma_{Yn}^2[\delta_{Yi}]_n}{[\tilde{K}_Y]_{nn}^2}\left[\frac{1}{2}[\delta_{Yi}]_n - r_{Yn} + [C_i'\tilde{K}_Z]_n\right]\right) \times T\Big\}.$$

From this we see that condition (16) is equivalent to $\lim_{T\to\infty}\mathbb{E}_0^Q[\exp(\phi(T))] = 0$.

Q.E.D.

D Proof of Proposition 6

The total returns process for stock i in $\mathcal{M}$ is given by

$$s_i(T) = s_i(t)\exp\left(\int_t^T\left[\delta_i(u)du + \frac{dS_i(u)}{S_i(u)} - \frac{1}{2}\frac{d[S_i](u)}{S_i^2(u)}\right]\right).$$

Recall that

$$S_i(t) = \exp\Big(a_i \times t - B_i'Y(t) - C_i'Z(t)\Big).$$

Let us define an $N+M$ dimensional vector b as follows

$$b_i \equiv \begin{bmatrix} B_i \\ C_i \end{bmatrix}.$$

An application of Ito's lemma allows us to conclude that

$$\frac{dS_i(t)}{S_i(t)} = a_i dt - B_i'dY(t) - C_i'dZ(t) + \frac{1}{2}\sum_{n=1}^{N+M}\sum_{m=1}^{N+M}[b_i]_n[b_i]_m d[X_n, X_m](t),$$

where $[\cdot,\cdot]$ denotes the quadratic covariation process (see Protter (1995)). Recall that $X(t)$ is the stack of $Y(t)$ and $Z(t)$. Using these dynamics we also find that the quadratic variation process for stock i is given by

$$d[S_i](t) = S_i^2(t)\left(\sum_{n=1}^{N+M}\sum_{m=1}^{N+M}[b_i]_n[b_i]_m d[X_n, X_m](t)\right).$$

Using these results, we can rewrite $s_i(T)$ as

$$\begin{aligned} s_i(T) &= s_i(t)\exp\left(\int_t^T \Big[(\delta_{0i}+a_i)du + \delta'_{Yi}Y(u)du - B'_i dY(u) - C'_i dZ(u)\Big]\right) \\ &= s_i(t)\exp\left((\delta_{0i}+a_i)(T-t) - B'_i(Y(T)-Y(t)) + \int_t^T \Big[\delta'_{Yi}Y(u)du - C'_i dZ(u)\Big]\right). \end{aligned}$$

From (36), notice that

$$C'\tilde{k}_Z = C'\tilde{K}_Z + \delta'_Y,$$

and hence for each i in $\mathcal{M}$

$$\int_t^T \Big[C'_i dZ(u) - \delta'_{Yi}Y(u)du\Big] = \int_t^T C'_i dz(u).$$

From this we see that

$$s_i(T) = s_i(t)\exp\Big((\delta_{0i}+a_i)(T-t) - B'_i(Y(T)-Y(t)) - C'_i(z(T)-z(t))\Big).$$

Since $s_i(0) = S_i(0)$, $z(0) = Z(0)$, and $S_i(0) = \exp(-B'_iY(0) - C'_iZ(0))$, we find that

$$s_i(T) = \exp\Big((\delta_{0i}+a_i)T - B'_iY(T) - C'_iz(T)\Big).$$

This completes the proof.

Q.E.D.

E Proof of Proposition 7

The dynamics of the log portfolio value are given by

$$d\log\omega(t) = \frac{d\omega(t)}{\omega(t)} - \frac{1}{2}\frac{d[\omega(t)]}{\omega(t)^2}.$$

From the portfolio dynamics in equation (40), we have that

$$d\log\omega(t) = \sum_{i=1}^{I} f_i(t)\left(\frac{dS_i(t)}{S_i(t)} + \delta_i(t)dt\right) - \frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{I} f_i(t)f_j(t)\frac{d[S_i,S_j]}{S_i(t)S_j(t)}.$$

According to (23)

$$S_i(t,T) = \exp\Big(a_i \times t - B_i'Y(t) - C_i'Z(t)\Big).$$

Therefore we have that

$$\frac{dS_i(t)}{S_i(t)} = a_i dt - b_i' dX(t) + \frac{1}{2}\mathrm{tr}\Big(b_i b_i' \Sigma_X V(t) \Sigma_X'\Big)$$

where

$$b_i \equiv \begin{bmatrix} B_i \\ C_i \end{bmatrix}$$

and $X(t) \equiv [Y(t)'Z(t)']'$. Also the quadratic covariation between stocks i and j is given by

$$\frac{d[S_i, S_j]}{S_i(t)S_j(t)} = \mathrm{tr}\Big(b_i b_j' \Sigma_X V(t) \Sigma_X'\Big).$$

We can therefore write the portfolio dynamics as

$$\begin{aligned} d\log\omega(t) &= \sum_{i=1}^{I} f_i(t)\Big(a_i dt - b_i' dX(t) + \delta_i(t)dt\Big) + \frac{1}{2}\sum_{i=1}^{I} f_i(t)\mathrm{tr}(b_i b_i' \Sigma_X V(t)\Sigma_X')dt \\ &\quad -\frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{I} f_i(t) f_j(t) \mathrm{tr}\Big(b_i b_j' \Sigma_X V(t) \Sigma_X'\Big)dt. \end{aligned} \tag{82}$$

We now make several observations:

$$\begin{aligned} b_i' dX(t) &= B_i' \tilde{K}_Y(\tilde{\Theta} - Y(t))dt + B_i'\Sigma_Y \sqrt{V(t)} d\tilde{W}(t) \\ &\quad + C_i'\tilde{\mu}dt - C_i'\tilde{K}_Z Y(t)dt + C_i'\Sigma_Z\sqrt{V(t)}d\tilde{W}(t), \\ \mathrm{tr}(b_i b_i' \Sigma_X V(t)\Sigma_X') &= \sum_{n=1}^{N+M} [b_i'\Sigma_X]_n^2 (\alpha_n + \beta_n' Y(t)), \\ \mathrm{tr}(b_i b_j' \Sigma_X V(t)\Sigma_X') &= \sum_{n=1}^{N+M} [b_i'\Sigma_X]_n \, [b_j'\Sigma_X]_n (\alpha_n + \beta_n' Y(t)). \end{aligned}$$

Using these we can rewrite (82) as

$$\begin{aligned} d\log\omega(t) &= \sum_{i=1}^{I} f_i(t)\tilde{a}_i(t)dt + \sum_{i=1}^{I} f_i(t)\delta_i(t)dt - \sum_{i=1}^{I} f_i(t) C_i' dZ(t) \\ &\quad -\sum_{i=1}^{I} f_i(t)\Big(B_i'\tilde{K}_Y\tilde{\Theta}dt + B_i'\Sigma_Y\sqrt{V(t)}d\tilde{W}(t)\Big) + \sum_{i=1}^{I} f_i(t) Y(t)' \tilde{K}_Y' \tilde{B}_i(t) dt, \end{aligned}$$

where we have that

$$\begin{aligned} \tilde{a}_i(t) &\equiv r_0 - \delta_{0i} + \tilde{\Theta}'\tilde{K}_Y' B_i + \tilde{\mu}' C_i - \frac{1}{2}\sum_{n=1}^{N+M}[b_i'\Sigma_X]_n \sum_{j=1}^{I} f_j(t)[b_j'\Sigma_X]_n \alpha_n, \\ \tilde{K}_Y'\tilde{B}_i(t) &\equiv r_Y - \delta_Y - \tilde{K}_Z' C_i - \frac{1}{2}\sum_{n=1}^{N+M}[b_i'\Sigma_X]_n \sum_{j=1}^{I} f_j(t)[b_j'\Sigma_X]_n \beta_n. \end{aligned}$$

With this, we can rewrite $d\log\omega(t)$ as

$$\begin{aligned} d\log\omega(t) &= \bar{\tilde{a}}(t)dt + \bar{\delta}(t)dt - \bar{\tilde{B}}(t)'dY(t) - \bar{C}(t)'dZ(t) \\ &+\left(\bar{\tilde{B}}(t) - \bar{B}(t)\right)'\left(\tilde{K}_Y\tilde{\Theta}dt + \Sigma_Y\sqrt{V(t)}d\tilde{W}(t)\right), \end{aligned}$$

where we have used the definition of $\bar{\cdot}$ as the averaging operator over $f(t)$. From the defitions of B_i from equation (9) and of $\tilde{B}_i(t)$ from above, we have that

$$\begin{aligned} \tilde{K}_Y'\bar{B}(t) &= r_Y - \bar{\delta}_Y - \tilde{K}_Z'\bar{C}(t) - \frac{1}{2}\sum_{n=1}^{N+M}\sum_{i=1}^{I} f_i(t)[b_i'\Sigma_X]_n^2\beta_n, \\ \tilde{K}_Y'\bar{\tilde{B}}(t) &= r_Y - \bar{\delta}_Y - \tilde{K}_Z'\bar{C}(t) - \frac{1}{2}\sum_{n=1}^{N+M}[\bar{b}(t)'\Sigma_X]_n^2\beta_n. \end{aligned}$$

From this it follows that

$$\begin{aligned} \bar{\tilde{B}}(t) - \bar{B}(t) &= \frac{1}{2}\left(\tilde{K}_Y'\right)^{-1}\left(\sum_{n=1}^{N+M}\left[\sum_{i=1}^{I} f_i(t)[b_i'\Sigma_X]_n^2 - [\bar{b}(t)'\Sigma_X]_n^2\right]\beta_n\right), \\ &= \frac{1}{2}\left(\tilde{K}_Y'\right)^{-1}\left(\sum_{n=1}^{N+M}[\Sigma_X]_n\left[\sum_{i=1}^{I} f_i(t)b_ib_i' - \bar{b}(t)\bar{b}(t)'\right][\Sigma_X]_n'\beta_n\right), \end{aligned}$$

where for a matrix $[\cdot]_n$ denotes the n^{th} row. Finally, using this we have that

$$\begin{aligned} d\log\omega(t) &= \bar{\tilde{a}}(t)dt + \bar{\delta}(t)dt - \bar{\tilde{B}}(t)'dY(t) - \bar{C}(t)'dZ(t) \\ &+\frac{1}{2}\left(\sum_{n=1}^{N+M}[\Sigma_X]_n\left[\sum_{i=1}^{I} f_i(t)b_ib_i' - \bar{b}(t)\bar{b}(t)'\right][\Sigma_X]_n'\beta_n'\right)\left(\tilde{K}_Y\right)^{-1}\times \\ &\left(\tilde{K}_Y\tilde{\Theta}dt + \Sigma_Y\sqrt{V(t)}d\tilde{W}(t)\right). \end{aligned}$$

This gives equation (43) the text.

Q.E.D.

F Proof of Proposition 9

Let us define the Ito operator $\mathcal{D}_\pi$ as

$$\mathcal{D}_\pi f(X,\xi) = \mathcal{D}_X f + \frac{1}{2}\frac{\partial^2 f}{\partial\xi^2}\sigma_D{}'\sigma_D + \sum_{n=1}^{N+M}\frac{\partial^2 f}{\partial\xi\partial X_n}\sigma_D{}'\sigma_{X_n} \tag{83}$$

where $\mathcal{D}_X$ is the Ito operator associated with X, and is given in equation (3), and where σ_{X_n} is given by

$$\sigma_{X_n}' = [\Sigma_X]_n\sqrt{V}.$$

The notation $[\cdot]_n$ indicates the n^{th} row of a matrix. We note that (66) is equivalent to $h(t)$ being a $\mathcal{Q}$ martingale. Applying Ito's lemma to $h(t)$ yields that

$$dh(t) = \left(\mathcal{D}_\pi h + \frac{\partial h}{\partial t}\right) dt + \frac{\partial h}{\partial \xi}\sigma_D{}' d\tilde{W}(t) + \frac{\partial h}{\partial X'}\Sigma_X \sqrt{V} d\tilde{W}(t).$$

The fact that (70) and (71) are necessary follows from the fact that Ito integrals are martingales only if they have a zero drift. The fact that (70) and (71) are sufficient is obvious.

For the second part of the proposition, we note that the existence of an admissible $\pi(t)$ implies that the following limit

$$\lim_{T\to\infty} \left\{ \begin{array}{c} \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^T \exp\left\{-\int_t^s \left(r - \tilde{\mu}_D + \frac{1}{2}\sigma_D'\sigma_D\right) du + \int_t^s \sigma_D' d\tilde{W}(u)\right\} ds\right] \\ +\mathbb{E}_t^{\mathcal{Q}} \left[\exp\left\{-\int_t^T \left(r - \tilde{\mu}_D + \frac{1}{2}\sigma_D'\sigma_D\right) du + \int_t^T \sigma_D' d\tilde{W}(u)\right\} \pi(T)\right] \end{array} \right\}$$

exists, and is equal to $\pi(t)$. Because the limit of the second expectation is equal to zero by the transversality condition (67), the limit of the first expectation must exist, and must be given by

$$\lim_{T\to\infty} \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^T \exp\left\{-\int_t^s \left(r - \tilde{\mu}_D + \frac{1}{2}\sigma_D'\sigma_D\right) du + \int_t^s \sigma_D' d\tilde{W}(u)\right\} ds\right] = \pi(t).$$

Because the term inside the expectation is positive, the monotone convergence theorem allows us to bring the limit inside the expectation, yielding that

$$\pi(t) = \mathbb{E}_t^{\mathcal{Q}} \left[\int_t^\infty \exp\left\{-\int_t^s \left(r - \tilde{\mu}_D + \frac{1}{2}\sigma_D'\sigma_D\right) du + \int_t^s \sigma_D' d\tilde{W}(u)\right\} ds\right].$$

This confirms the claim that an admissible price to dividend process is unique, as well as the relationship in (72).

Q.E.D.

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