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Portfolio Diversification and Value At Risk Under Thick-Tailendness

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PORTFOLIO DIVERSIFICATION AND VALUE AT RISK UNDER THICK-TAILEDNESS[1]

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ABSTRACT

We present a unified approach to value at risk analysis under heavy-tailedness using new majorization theory for linear combinations of thick-tailed random variables that we develop. Among other results, we show that the stylized fact that portfolio diversification is always preferable is reversed for extremely heavy-tailed risks or returns. The stylized facts on diversification are nevertheless robust to thick-tailedness of risks or returns as long as their distributions are not extremely long-tailed. We further demonstrate that the value at risk is a coherent measure of risk if distributions of risks are not extremely heavy-tailed. However, coherency of the value at risk is always violated under extreme thick-tailedness. Extensions of the results to the case of dependence, including convolutions of $\alpha-$symmetric distributions and models with common shocks are provided.

KEYWORDS: value at risk, coherent measures of risk, heavy-tailed risks, portfolios, riskiness, diversification, risk bounds

JEL Classification: G11

1 Introduction

1.1 Objectives and key results

Value at risk (VaR) models are examples of many models in economics, finance and risk management that have a structure that depends on majorization phenomena for linear combinations of random variables (r.v.'s). The majorization relation is a formalization of the concept of diversity in the components of vectors. Over the past decades, majorization theory, which focuses on the study of the majorization ordering and functions that preserve it, has found applications in disciplines ranging from statistics, probability theory and economics to mathematical genetics, linear algebra and geometry.

This paper presents a new unified framework for portfolio value at risk analysis under thick-tailedness assumptions using new majorization theory for linear combinations of heavy-tailed r.v.'s that we develop.

We provide a precise formalization of the concept of portfolio diversification on the basis of majorization ordering (see Section 3). We further show, for the first time in the literature, that the stylized fact that portfolio diversification is always preferable is reversed for a wide class of distributions of risks (Theorem 4.2). The class of distributions for which this is the case is the class of extremely heavy-tailed distributions: a diversification of a portfolio of extremely thick-tailed risks always leads to an increase in the riskiness of their portfolio. The encouraging message of the results obtained in this paper is that the stylized facts on diversification are nevertheless robust to thick-tailedness of risks or returns as long as their distributions are not extremely long-tailed (Theorem 4.1).

Moreover, we demonstrate that, in the world of not extremely heavy-tailed risks, VaR exhibits the property of subadditivity and thus satisfies the important condition of coherency, which is a natural requirement to be imposed on a measure of risk from the points of view of exchange, regulators and society (Theorem 5.1). However, coherency of the value at risk is violated, even in the case of independence, if distributions of risks are extremely thick-tailed (Theorem 5.2). One should indicate here that, so far, only a few particular counterexamples that show that VaR is not, in general, a coherent measure of risk were available in the literature (see Artzner, Delbaen, Eber and Heath, 1999, and Embrechts, McNeil and Straumann, 2002). Our results demonstrate, on the other hand, that the value at risk *always* lacks the coherency property for a wide class of risks with extremely heavy-tailed distributions. We also obtain sharp bounds on the VaR of the returns on portfolios of risks with long-tailed returns (Theorems 4.3 and 4.4).

In other words, according to our results, the stylized facts on portfolio diversification and value at risk coherency are robust to the assumptions of heavy-tailedness of distributions of risks if distributions entering these assumptions are not extremely thick-tailed. The stylized facts and VaR coherency are not robust to distributional assumptions involving extremely heavy-tailed distributions.

Furthermore, we obtain extensions of the above results for a wide class of dependent risks (see Section 6). Namely, we show all the results in the paper continue to hold for convolutions of dependent risks with joint $\alpha-$symmetric distributions and their analogues with non-identical marginals.[1] The class of $\alpha-$symmetric distributions is very

[1] An $n-$dimensional distribution is called $\alpha-$symmetric if its characteristic function can be written as $\phi((\sum_{i=1}^{n}|t_i|^{\alpha})^{1/\alpha})$, where ϕ is

wide and includes, in particular, spherical distributions corresponding to $\alpha = 2$. Important examples of spherical distributions, in turn, are given by Kotz type, multinormal and logistic distributions and multivariate stable laws. In addition, they include a subclass of mixtures of normal distributions as well as multivariate $t-$distributions that were used in a number of papers to model heavy-tailedness phenomena with dependence and finite moments up to a certain order (see, among others, Praetz, 1972, Blattberg and Gonedes, 1974, and Glasserman, Heidelberger and Shahabuddin, 2002). Moreover, the class of $\alpha-$symmetric distributions includes a wide class of convolutions of models with common shocks affecting all risks (such as macroeconomic or political ones, see Andrews, 2003) which are of great importance in economics and finance.

1.2 Heavy-tailedness in economic and financial data and its modelling

This paper belongs to a large stream of literature in economics and finance that have focused on the analysis of thick-tailed phenomena. This stream of literature goes back to Mandelbrot (1963) (see also the papers in Mandelbrot, 1997, and Fama, 1965), who pioneered the study of heavy-tailed distributions with tails declining as $x^{-\alpha}$, $\alpha > 0$, in these fields. If a model involves a r.v. X with such thick-tailed distribution, then

$$P(|X| > x) \sim x^{-\alpha}. \tag{1}$$

The r.v. X for which this is the case has finite moments $E|X|^p$ of order $p < \alpha$. However, the moments are infinite for $p \geq \alpha$.

It was documented in numerous studies that the time series encountered in many fields in economics and finance are heavy-tailed (see the discussion in Loretan and Phillips, 1994, Meerschaert and Scheffler, 2000, Gabaix, Gopikrishnan, Plerou and Stanley, 2003, and references therein). Motivated by these empirical findings, a number of studies in financial economics have focused on portfolio and value-at-risk modelling with heavy-tailed returns (see, e.g., the reviews in Duffie and Pan, 1997, Uchaikin and Zolotarev, 1999, Ch. 17, and Glasserman, Heidelberger and Shahabuddin (2002)). Several authors considered problems of statistical inference for data from thick-tailed populations (see Loretan and Phillips, 1994, the papers in Adler, Feldman and Taqqu, 1998, and references therein).

Mandelbrot (1963) presented evidence that historical daily changes of cotton prices have the tail index $\alpha \approx 1.7$, and thus have infinite variances. Using different models and statistical techniques, subsequent research reported the following estimates of the tail parameters α for returns on various stocks and stock indices:

$3 < \alpha < 5$ (Jansen and de Vries, 1991),

$2 < \alpha < 4$ (Loretan and Phillips, 1994),

$1.5 < \alpha < 2$ (McCulloch, 1996, 1997),

$0.9 < \alpha < 2$ (Rachev and Mittnik, 2000).

Recent studies (see Gabaix et. al., 2003, and references therein) have found that the returns on many stocks

a continuous function and $\alpha > 0$. Such distributions should not be confused with multivariate spherically symmetric stable distributions, which have characteristic functions $\exp\big[-\lambda\big(\sum_{i=1}^n t_i^2\big)^{\beta/2}\big]$, $0 < \beta \leq 2$. Obviously, spherically symmetric stable distributions are particular examples of $\alpha-$symmetric distributions with $\alpha = 2$ (that is, of spherical distributions) and $\phi(x) = \exp(-x^\beta)$.

and stock indices have the tail exponent $\alpha \approx 3$, while the distributions of trading volume and the number of trades on financial markets obey the power laws (1) with $\alpha \approx 1.5$ and $\alpha \approx 3.4$, respectively. As discussed in Gabaix et. al. (2003), these estimates of the tail indices α are robust to different types and sizes of financial markets, market trends and are similar for different countries. Motivated by these empirical findings, Gabaix et. al. (2003) proposed a model that demonstrated that the above power laws for stock returns, trading volume and the number of trades are explained by trading of large market participants, namely, the largest mutual funds whose sizes have the tail exponent $\alpha \approx 1$. Power laws (1) with $\alpha \approx 1$ (Zipf laws) have also been found to hold for firm sizes (see Axtell, 2001) and city sizes (see Gabaix, 1999a, b for the discussion and explanations of the Zipf law for cities). One should also note that some studies also report the tail exponent to be close to one or even slightly less than one for such financial time series as Bulgarian lev/US dollar exchange spot rates and increments of the market time process for Deutsche Bank price record (see Rachev and Mittnik, 2000).

The fact that a number of economic and financial time series have the tail exponents of approximately one is very important in the context of the results in this paper: as we demonstrate, the conclusions of portfolio value at risk theory for risk distributions with the tail exponents $\alpha < 1$ with infinite means are the opposites of those for distributions with $\alpha > 1$ for which the first moment is finite.

Several frameworks have been proposed to model heavy-tailedness phenomena, including stable distributions, Pareto distributions, multivariate $t-$distributions, mixtures of normals, power exponential distributions, ARCH processes, mixed diffusion jump processes, variance gamma and normal inverse Gamma distributions. However, the debate concerning the values of the tail indices for different heavy-tailed financial data and on appropriateness of their modelling based on certain above distributions is still under way in empirical literature. In particular, as indicated before, a number of studies continue to find tail parameters less than two in different financial data sets and also argue that stable distributions are appropriate for their modelling.

1.3 Thick tails and extremely thick tails and extensions to the case of dependence

To illustrate the main ideas of the proof and in order to simplify the presentation of the main results in this paper, we first model heavy-tailedness using the framework of independent stable distributions and their convolutions. More precisely, the class of not extremely thick-tailed distributions is first modelled using convolutions of stable distributions with (different) indices of stability greater than one. Similarly, the results of the paper for extremely heavy-tailed case are first presented and proven using the framework of convolutions of stable distributions with characteristic exponents less than one. The former class has tail exponents $\alpha > 1$ (and thus, as discussed above, the stylized facts on portfolio diversification and value at risk coherency hold for risk distributions from this class) and for the latter class one has $\alpha < 1$ (so that the stylized facts on portfolio diversification exhibit reversals and the VaR is not a coherent risk measure in the world of risk with distributions from this class). In some places throughout paper, we will omit the words "not extremely" in the discussion of the results for distributions with $\alpha > 1$ (or with relatively large α) and refer to such distributions as just "heavy-tailed" or "thick-tailed", if this does not lead to a confusion. The class of "not extremely heavy-tailed" or, with this convention on the terminology, of "heavy-tailed" distributions is thus opposed to the class of "extremely thick-tailed" distributions with $\alpha < 1$ (or with relatively small α).

In Section 6 we show, however, that all the results obtained in this paper continue to hold for a wide class of multivariate distributions for which marginals are dependent and can be non-identical and, in addition to that, can have finite variances, unlike stable distributions and their convolutions. As indicated before, according to these extensions, all the results in the paper continue to hold for convolutions of $\alpha-$symmetric distributions and their analogues with non-identical one-dimensional marginals (see Subsection 1.1). Similar to the framework based on stable distributions, the stylized facts on portfolio diversification hold and the value at risk exhibits coherency for convolutions of $\alpha-$symmetric distributions with $\alpha > 1$. The stylized facts are reversed and VaR coherency is violated in the case of convolutions of $\alpha-$symmetric distributions with $\alpha < 1$.

One should also note here that all the results in the paper are available for the case of skewed distributions (see Remark 4.3), including skewed stable distributions (such as, for instance, extremely heavy-tailed Lévy distributions with $\alpha = 1/2$ concentrated on the positive semi-axis) and, according to the extensions discussed above, $\alpha-$symmetric distributions with skewed marginals. Therefore, this paper, in fact, succeeds in the unification of the robustness of majorization properties of convolutions of distributions and their implications for portfolio value at risk theory to all the main distributional properties: heavy-tailedness, dependence, skewness and the case of non-identical one-dimensional distributions.

1.4 Optimistic implications

Besides portfolio value at risk theory under heavy-tailedness dealt with in this paper, our results on portfolio VaR comparisons and their analogues on majorization properties of tail probabilities of linear combinations of r.v.'s have many other applications. These applications, presented, for the most part, in the author's Ph.D. dissertation Ibragimov (2005), include the study of efficiency of linear estimators and monotone consistency of the sample mean, robustness of the model of demand-driven innovation and spatial competition over time, value at risk analysis, optimal strategies for a multiproduct monopolist as well that of inheritance models in mathematical evolutionary theory.[2]

[2]The following list summarizes some of these applications.

(i) From the analogues of the results on portfolio diversification and VaR analysis and their analogues on majorization properties of tail probabilities of linear combinations of heavy-tailed r.v.'s it follows that the sample mean is the best linear unbiased estimator of the population mean for not extremely heavy-tailed populations in the sense of its peakedness properties. Moreover, in such a case, the sample mean exhibits the important property of monotone consistency and, thus, an increase in the sample size always improves its performance. However, efficiency of the sample mean in the sense of its peakedness decreases with the sample size if the sample mean is used to estimate the population center under extreme thick-tailedness. The majorization results also provide sharp concentration inequalities for linear estimators as well as their extensions to the case of wide classes of dependent data.

(ii) We develop a framework that allows one to model the optimal bundling problem of a multiproduct monopolist providing interrelated goods with an arbitrary degree of complementarity or substitutability. Characterizations of optimal bundling strategies are derived for the seller in the case of long-tailed valuations and tastes for the products. We show, in particular, that if goods provided in a Vickrey auction or any other revenue equivalent auction are substitutes and bidders' tastes for the objects are not extremely heavy-tailed, then the monopolist prefers separate provision of the products. However, if the goods are complements and consumers' tastes are extremely thick-tailed, then the seller prefers providing the products on a single auction. We also present results on consumers' preferences over bundled auctions in the case when their valuations exhibit heavy-tailedness. In addition, we obtain characterizations of optimal bundling strategies for a monopolist who provides complements or substitutes for profit-maximizing prices to buyers with long-tailed tastes.

(iii) Another application of the main majorization results explored in depth in Ibragimov (2005) concerns the analysis of growth of firms that invest into learning about the next period's optimal product. We present a study of robustness of the model of demand-driven innovation and spatial competition over time with log-concavely distributed signals developed by Jovanovic and Rob (1987) to heavy-tailedness assumptions. The implications of the model remain valid for not extremely long-tailed distributions of consumers' signals. However, again these properties are reversed for signals with extremely thick-tailed densities.

The main message of the results in this paper and of other their applications is that the presence of heavy-tailedness can either reinforce or reverse the implications of models in economics, finance and risk management, depending on the degree of thick-tailedness. Similar to the properties of the value at risk derived in this paper, the standard implications of models in the above fields continue to hold for not extremely heavy-tailed distributions. However, these properties are reversed under the assumptions of extreme thick-tailedness.

This message is optimistic since, as discussed before, many economic models are robust to heavy-tailedness (and dependence) as long as the tail indices $\alpha > 1$ and empirical studies observe such values for α in most of economic and financial time series. However, the reversals of the models are possible for a wide class of extremely thick-tailed distributions. Therefore, the models should be applied with care in presence of very heavy-tailed signals, especially in the case of the tail indices close to the critical boundary $\alpha = 1$.

1.5 Organization of the paper

The paper is organized as follows: Section 2 contains notations and definitions of classes of heavy-tailed distributions used throughout the paper and reviews their properties. Section 3 discusses the definition of majorization ordering and introduces the formalization of the concept of portfolio diversification on its basis. In Section 4, we present the main results of the paper on the effects of diversification of a portfolio on its riskiness. Section 5 contains the main results of the paper on (non)coherency properties of the VaR under thick-tailedness. Sections 6 and 7 discuss extensions of the results in the paper to the case of dependence, including convolutions of α-symmetric and spherical distributions and models with common shocks, and make some concluding remarks. Finally, Section 8 contains proofs of the results obtained in the paper.

2 Notations

In this section, we introduce classes of distributions we will be dealing with throughout the paper.

We say that a r.v. X with density $f : \mathbf{R} \to \mathbf{R}$ and the convex distribution support $\Omega = \{x \in \mathbf{R} : f(x) > 0\}$ is log-concavely distributed if $log\ f(x)$ is concave in $x \in \Omega$, that is, if for all $x_1, x_2 \in \Omega$, and any $\lambda \in [0, 1]$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq (f(x_1))^{\lambda}(f(x_2))^{1-\lambda}. \tag{2}$$

(see An, 1998). A distribution is said to be log-concave if its density f satisfies (2).

Log-concave distributions have many appealing properties that have been utilized in a number of works in economics and finance (see the surveys in Karlin, 1968, Marshall and Olkin, 1979, and An, 1998).[3] However, such

(iv) We study transmission of traits through generations in multifactorial inheritance models with sex- and time-dependent heritability. We further analyze the implications of these models under heavy-tailedness of traits' distributions. Among other results, we show that in the case of a trait (for instance, a medical or behavioral disorder or a phenotype with significant heritability affecting human capital in an economy) with not very thick-tailed initial density, the trait distribution becomes increasingly more peaked, that is, increasingly more concentrated and unequally spread, with time. But these patterns are reversed for traits with sufficiently heavy-tailed initial distributions (e.g., a medical or behavioral disorder for which there is no strongly expressed risk group or a relatively equally distributed ability with significant genetic influence). Such traits' distributions become less peaked over time and increasingly more spread in the population.

[3] Some of these properties are the following:

distributions cannot be used in the study of thick-tailedness phenomena since any log-concave density is extremely light-tailed: in particular, if a r.v. X is log-concavely distributed, then its density has at most an exponential tail, that is, $f(x) = o(exp(-\lambda x))$ for some $\lambda > 0$, as $x \to \infty$ and all the power moments $E|X|^\gamma$, $\gamma > 0$, of the r.v. exist (see Corollary 1 in An, 1998).

Throughout the paper, $\mathcal{LC}$ denotes the class of symmetric log-concave distributions.[4]

For $0 < \alpha \leq 2$, $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbf{R}$, we denote by $S_\alpha(\sigma, \beta, \mu)$ the stable distribution with the characteristic exponent (index of stability) α, the scale parameter σ, the symmetry index (skewness parameter) β and the location parameter μ. That is, $S_\alpha(\sigma, \beta, \mu)$ is the distribution of a r.v. X with the characteristic function

$$E(e^{ixX}) = \begin{cases} exp\left\{i\mu x - \sigma^\alpha |x|^\alpha (1 - i\beta sign(x) tan(\pi\alpha/2))\right\}, & \\ & \alpha \neq 1, \\ exp\left\{i\mu x - \sigma|x|(1 + (2/\pi) i\beta sign(x) ln|x|\right\}, & \alpha = 1, \end{cases}$$

$x \in \mathbf{R}$, where $i^2 = -1$ and $sign(x)$ is the sign of x defined by $sign(x) = 1$ if $x > 0$, $sign(0) = 0$ and $sign(x) = -1$ otherwise. In what follows, we write $X \sim S_\alpha(\sigma, \beta, \mu)$, if the r.v. X has the stable distribution $S_\alpha(\sigma, \beta, \mu)$.

As is well-known, a closed form expression for the density $f(x)$ of the distribution $S_\alpha(\sigma, \beta, \mu)$ is available in the following cases (and only in those cases): $\alpha = 2$ (Gaussian distributions); $\alpha = 1$ and $\beta = 0$ (Cauchy distributions); $\alpha = 1/2$ and $\beta \pm 1$ (Lévy distributions).[5] Degenerate distributions correspond to the limiting case $\alpha = 0$.

The index of stability α characterizes the heaviness (the rate of decay) of the tails of stable distributions $S_\alpha(\sigma, \beta, \mu)$. In particular, if $X \sim S_\alpha(\sigma, \beta, \mu)$, then its distribution satisfies power law (1). This implies that the $p-$th absolute moments $E|X|^p$ of a r.v. $X \sim S_\alpha(\sigma, \beta, \mu)$, $\alpha \in (0, 2)$ are finite if $p < \alpha$ and are infinite otherwise.

The symmetry index β characterizes the skewness of the distribution. The stable distributions with $\beta = 0$ are symmetric about the location parameter μ. The stable distributions with $\beta = \pm 1$ and $\alpha \in (0, 1)$ (and only they) are one-sided, the support of these distributions is the semi-axis $[\mu, \infty)$ for $\beta = 1$ and is $(-\infty, \mu]$ (in particular, the Lévy distribution with $\mu = 0$ is concentrated on the positive semi-axis for $\beta = 1$ and on the negative semi-axis for $\beta = -1$). In the case $\alpha > 1$ the location parameter μ is the mean of the distribution $S_\alpha(\sigma, \beta, \mu)$. The scale parameter σ is a generalization of the concept of standard deviation; it coincides with the standard deviation in the special case of Gaussian distributions ($\alpha = 2$).

Distributions $S_\alpha(\sigma, \beta, \mu)$ with $\mu = 0$ for $\alpha \neq 1$ and $\beta \neq 0$ for $\alpha = 1$ are called strictly stable. If $X_i \sim S_\alpha(\sigma, \beta, \mu)$, $\alpha \in (0, 2]$, are i.i.d. strictly stable r.v.'s, then, for all $a_i \geq 0$, $i = 1, ..., n$, $\sum_{i=1}^n a_i X_i / \left(\sum_{i=1}^n a_i^\alpha\right)^{1/\alpha} \sim S_\alpha(\sigma, \beta, \mu)$.

For a detailed review of properties of stable distributions the reader is referred to, e.g., the monographs by

Any log-concave density is unimodal. Moreover, it has the property of strong unimodality, that is, its convolution with any other unimodal density is again unimodal;

The survivor and distribution functions of log-concave densities are both log-concave and, thus, a log-concavely distributed r.v. has the new-better-than-used property;

A log-concave density is of Pólya frequency of order 2 (PF-2);

The hazard function of a log-concave density is monotonically increasing.

Examples of log-concave distributions include the normal distribution, the uniform density, the exponential density, the Gamma distribution $\Gamma(\alpha, \beta)$ with the shape parameter $\alpha \geq 1$, the Beta distribution $\mathcal{B}(a, b)$ with $a \geq 1$ and $b \geq 1$; the Weibull distribution $\mathcal{W}(\gamma, \alpha)$ with the shape parameter $\alpha \geq 1$.

[4] $\mathcal{LC}$ stands for "log-concave".

[5] The densities of Cauchy distributions are $f(x) = \sigma/(\pi(\sigma^2 + (x - \mu)^2))$; Lévy distributions have densities $f(x) = (\sigma/(2\pi))^{1/2} exp(-\sigma/(2x)) x^{-3/2}$, $x \geq 0$; $f(x) = 0$, $x < 0$, where $\sigma > 0$, and their shifted versions.

Zolotarev (1986) and Uchaikin and Zolotarev (1999).

For $0 < r < 2$, we denote by $\overline{\mathcal{CS}}(r)$ the class of distributions which are convolutions of symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with characteristic exponents $\alpha \in (r, 2]$ and $\sigma > 0$.[6] That is, $\overline{\mathcal{CS}}(r)$ consists of distributions of r.v.'s X such that, for some $k \geq 1$, $X = Y_1 + ... + Y_k$, where Y_i, $i = 1, ..., k$, are independent r.v.'s such that $Y_i \sim S_{\alpha_i}(\sigma_i, 0, 0)$, $\alpha_i \in (r, 2]$, $\sigma_i > 0$, $i = 1, ..., k$.

Further, for $0 < r \leq 2$, $\underline{\mathcal{CS}}(r)$ stands for the class of distributions which are convolutions of symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with indices of stability $\alpha \in (0, r)$ and $\sigma > 0$.[7] That is, $\underline{\mathcal{CS}}(r)$ consists of distributions of r.v.'s X such that, for some $k \geq 1$, $X = Y_1 + ... + Y_k$, where Y_i, $i = 1, ..., k$, are independent r.v.'s such that $Y_i \sim S_{\alpha_i}(\sigma_i, 0, 0)$, $\alpha_i \in (0, r)$, $\sigma_i > 0$, $i = 1, ..., k$.

Finally, we denote by $\overline{\mathcal{CSLC}}$ the class of convolutions of distributions from the classes $\mathcal{LC}$ and $\overline{\mathcal{CS}}(1)$. That is, $\overline{\mathcal{CSLC}}$ is the class of convolutions of symmetric distributions which are either log-concave or stable with characteristic exponents greater than one.[8] In other words, $\overline{\mathcal{CSLC}}$ consists of distributions of r.v.'s X such that $X = Y_1 + Y_2$, where Y_1 and Y_2 are independent r.v.'s with distributions belonging to $\mathcal{LC}$ or $\overline{\mathcal{CS}}(1)$.

All the classes $\mathcal{LC}$, $\overline{\mathcal{CSLC}}$, $\overline{\mathcal{CS}}(r)$ and $\underline{\mathcal{CS}}(r)$ are closed under convolutions. In particular, the class $\overline{\mathcal{CSLC}}$ coincides with the class of distributions of r.v.'s X such that, for some $k \geq 1$, $X = Y_1 + ... + Y_k$, where Y_i, $i = 1, ..., k$, are independent r.v.'s with distributions belonging to $\mathcal{LC}$ or $\overline{\mathcal{CS}}(1)$.

A linear combination of independent stable r.v.'s with the *same* characteristic exponent α also has a stable distribution with the same α. However, in general, this does not hold in the case of convolutions of stable distributions with *different* indices of stability. Therefore, the class $\overline{\mathcal{CS}}(r)$ of *convolutions* of symmetric stable distributions with *different* indices of stability $\alpha \in (r, 2]$ is wider than the class of *all* symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with $\alpha \in (r, 2]$ and $\sigma > 0$. Similarly, the class $\underline{\mathcal{CS}}(r)$ is wider than the class of *all* symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with $\alpha \in (0, r)$ and $\sigma > 0$.

Clearly, $\overline{\mathcal{CS}}(1) \subset \overline{\mathcal{CSLC}}$ and $\mathcal{LC} \subset \overline{\mathcal{CSLC}}$. It should also be noted that the class $\overline{\mathcal{CSLC}}$ is wider than the class of (two-fold) convolutions of log-concave distributions with stable distributions $S_\alpha(\sigma, 0, 0)$ with $\alpha \in (1, 2]$ and $\sigma > 0$.

By definition, for $0 < r_1 < r_2 \leq 2$, the following inclusions hold: $\overline{\mathcal{CS}}(r_2) \subset \overline{\mathcal{CS}}(r_1)$ and $\underline{\mathcal{CS}}(r_1) \subset \underline{\mathcal{CS}}(r_2)$.

In some sense, symmetric (about $\mu = 0$) Cauchy distributions $S_1(\sigma, 0, 0)$ are at the dividing boundary between the classes $\underline{\mathcal{CS}}(1)$ and $\overline{\mathcal{CS}}(1)$ (and between the classes $\underline{\mathcal{CS}}(1)$ and $\overline{\mathcal{CSLC}}$). Similarly, for $r \in (0, 2)$, symmetric stable distributions $S_r(\sigma, 0, 0)$ with the characteristic exponent $\alpha = r$ are at the dividing boundary between the classes $\underline{\mathcal{CS}}(r)$ and $\overline{\mathcal{CS}}(r)$. Further, symmetric normal distributions $S_2(\sigma, 0, 0)$ are at the dividing boundary between the class $\mathcal{LC}$ of log-concave distributions and the class $\underline{\mathcal{CS}}(2)$ of convolutions of symmetric stable distributions with indices of stability $\alpha < 2$.[9]

[6] Here and below, $\mathcal{CS}$ stands for "convolutions of stable"; the overline indicates that convolutions of stable distributions with indices of stability *greater* than the threshold value r are taken.

[7] The underline indicates considering stable distributions with indices of stability *less* than the threshold value r.

[8] $\mathcal{CSLC}$ is the abbreviation of "convolutions of stable and log-concave".

[9] More precisely, the symmetric Cauchy distributions are the only ones that belong to all the classes $\underline{\mathcal{CS}}(r)$ with $r > 1$ and all the classes $\overline{\mathcal{CS}}(r)$ with $r < 1$. Symmetric stable distributions $S_r(\sigma, 0, 0)$ are the only ones that belong to all the classes $\underline{\mathcal{CS}}(r')$ with $r' > r$ and all the classes $\overline{\mathcal{CS}}(r')$ with $r' < r$. Symmetric normal distributions are the only distributions belonging to the class $\mathcal{LC}$ and all the

In what follows, we write $X \sim \mathcal{LC}$ (resp., $X \sim \overline{\mathcal{CSLC}}$, $X \sim \overline{\mathcal{CS}}(r)$ or $X \sim \underline{\mathcal{CS}}(r)$) if the distribution of the r.v. X belongs to the class $\mathcal{LC}$ (resp., $\overline{\mathcal{CSLC}}$, $\overline{\mathcal{CS}}(r)$ or $\underline{\mathcal{CS}}(r)$).

3 Majorization and portfolio diversification

The present paper demonstrates that powerful tools for portfolio value at risk analysis are given by majorization theory. A vector $a \in \mathbf{R}^n$ is said to be majorized by a vector $b \in \mathbf{R}^n$, written $a \prec b$, if $\sum_{i=1}^k a_{[i]} \leq \sum_{i=1}^k b_{[i]}$, $k = 1, ..., n-1$, and $\sum_{i=1}^n a_{[i]} = \sum_{i=1}^n b_{[i]}$, where $a_{[1]} \geq \ldots \geq a_{[n]}$ and $b_{[1]} \geq \ldots \geq b_{[n]}$ denote components of a and b in decreasing order. The relation $a \prec b$ implies that the components of the vector a are more diverse than those of b (see Marshall and Olkin, 1979). In this context, it is easy to see that the following relations hold:

$$\Big(\sum_{i=1}^n a_i/n, ..., \sum_{i=1}^n a_i/n\Big) \prec (a_1, ..., a_n) \prec \Big(\sum_{i=1}^n a_i, 0, ..., 0\Big), \quad a \in \mathbf{R}_+^n, \tag{3}$$

for all $a \in \mathbf{R}_+^n$. In particular,

$$(1/(n+1), ..., 1/(n+1), 1/(n+1)) \prec (1/n, ..., 1/n, 0), \quad n \geq 1. \tag{4}$$

A function $\phi : A \to \mathbf{R}$ defined on $A \subseteq \mathbf{R}^n$ is called *Schur-convex* (resp., *Schur-concave*) on A if $(a \prec b) \Longrightarrow (\phi(a) \leq \phi(b))$ (resp. $(a \prec b) \Longrightarrow (\phi(a) \geq \phi(b))$ for all $a, b \in A$. If, in addition, $\phi(a) < \phi(b)$ (resp., $\phi(a) > \phi(b)$) whenever $a \prec b$ and a is not a permutation of b, then ϕ is said to be *strictly* Schur-convex (resp., *strictly* Schur-concave) on A.

In what follows, given a loss probability $q \in (0, 1/2)$ and a r.v. (risk) Z, we denote by $VaR_q(Z)$ the value at risk (VaR) of Z at level q, that is, its $(1-q)-$quantile.[10]

Throughout the paper, $\mathbf{R}_+$ stands for $\mathbf{R}_+ = [0, \infty)$. For $w = (w_1, ..., w_n) \in \mathbf{R}_+^n$, denote by Z_w the return on the portfolio of risks $X_1, ..., X_n$ with weights w. Most of the results in the paper do not require the assumption that $\sum_{i=1}^n w_i = 1$ for the portfolio weights w_i, $i = 1, ..., n$. If this the case, we write that w belongs to the simplex $\mathcal{I}_n = \{w = (w_1, ..., w_n) : w_i \geq 0, i = 1, ..., n, \sum_{i=1}^n w_i = 1\} : w \in \mathcal{I}_n$.

Denote $\underline{w} = (1/n, 1/n, ..., 1/n) \in \mathcal{I}_n$ and $\overline{w} = (1, 0, ..., 0) \in \mathcal{I}_n$. The expressions $VaR_q(Z_{\underline{w}})$ and $VaR_q(Z_{\overline{w}})$ are, thus, the values at risk of the portfolio with equal weights and of the portfolio consisting of only one return (risk).

Suppose that $v = (v_1, ..., v_n) \in \mathbf{R}_+^n$ and $w = (w_1, ..., w_n) \in \mathbf{R}_+^n$, $\sum_{i=1}^n v_i = \sum_{i=1}^n w_i$, are the weights of two portfolios of risks (or assets' returns). If $v \prec w$, it is natural to think about the portfolio with weights v as being more diversified than that with weights w so that, for example, the portfolio with equal weights $\underline{w}$ is the most diversified and the portfolio with weights $\overline{w}$ consisting of one risk is the least diversified among all the portfolios with weights $w \in \mathcal{I}_n$ (in this regard, the notion of one portfolio being more or less diversified than another one is, in some sense, the opposite of that for vectors of weights for the portfolio).

classes $\overline{\mathcal{CS}}(r)$ with $r \in (0, 2)$.

[10]That is, in the case of an absolutely continuous risk Z, $P(Z > VaR_q(Z)) = q$.

4 Main results: portfolio diversification under heavy-tailedness

In this section, we present the main results of the paper on portfolio diversification under heavy-tailedness.

According to the following theorem, diversification of a portfolio of not extremely thick-tailed risks X_i, $i = 1, ..., n$, with weights $w = (w_1, ..., w_n) \in \mathbf{R}_+^n$, leads to a decrease in the riskiness of its return $Z_w = \sum_{i=1}^n w_i X_i$ in the sense of (first-order) stochastic dominance.

Theorem 4.1 *Let $q \in (0, 1/2)$ and let X_i, $i = 1, ..., n$, be i.i.d. risks such that $X_i \sim \overline{\mathcal{CSLC}}$, $i = 1, ..., n$. Then*

(i) $VaR_q(Z_v) < VaR_q(Z_w)$ if $v \prec w$ and v is not a permutation of w (in other words, the function $\psi(w, q) = VaR_q(Z_w)$ is strictly Schur-convex in $w \in \mathbf{R}_+^n$).

(ii) In particular, $VaR_q(Z_{\underline{w}}) < VaR_q(Z_w) < VaR_q(Z_{\overline{w}})$ for all $q \in (0, 1/2)$ and all weights $w \in \mathcal{I}_n$ such that $w \neq \underline{w}$ and w is not a permutation of $\overline{w}$.

The results for portfolio VaR given by Theorem 4.1 are reversed under the assumption that the distributions of the risks $X_1, ..., X_n$ are extremely long-tailed. In such a setting, diversification of a portfolio of the risks increases riskiness of its return. We have the following

Theorem 4.2 *Let $q \in (0, 1/2)$ and let X_i, $i = 1, ..., n$, be i.i.d. risks such that $X_i \sim \underline{\mathcal{CS}}(1)$, $i = 1, ..., n$. Then*

(i) $VaR_q(Z_v) > VaR_q(Z_w)$ if $v \prec w$ and v is not a permutation of w (in other words, the function $\psi(w, q) = VaR_q(Z_w)$, is strictly Schur-concave in $w \in \mathbf{R}_+^n$).

(ii) In particular, $VaR_q(Z_{\overline{w}}) < VaR_q(Z_w) < VaR_q(Z_{\underline{w}})$ for all $q \in (0, 1/2)$ and all weights $w \in \mathcal{I}_n$ such that $w \neq \underline{w}$ and w is not a permutation of $\overline{w}$.

The following theorems give sharp bounds on the value at risk of portfolios of heavy-tailed returns (risks). These bounds refine and complement the estimates given by Theorems 4.1 and 4.2 in the world of heavy-tailed risks.

Theorem 4.3 *Let $q \in (0, 1/2)$, $r \in (0, 2)$ and let $X_1, ..., X_n$ be i.i.d. risks such that $X_i \sim \overline{\mathcal{CS}}(r)$, $i = 1, ..., n$. Then*

(i) $VaR_q(Z_v) < VaR_q(Z_w)$ if $(v_1^r, ..., v_n^r) \prec (w_1^r, ..., w_n^r)$ and $(v_1^r, ..., v_n^r)$ is not a permutation of $(w_1^r, ..., w_n^r)$ (that is, the function $\psi(w, q) = VaR_q(Z_w)$, $w \in \mathbf{R}_+^n$, is strictly Schur-convex in $(w_1^r, ..., w_n^r)$).

(ii) The following sharp bounds hold:

$$n^{1-1/r}(\sum_{i=1}^n w_i^r)^{1/r} VaR_q(Z_{\underline{w}}) < VaR_q(Z_w) < (\sum_{i=1}^n w_i^r)^{1/r} VaR_q(Z_{\overline{w}})$$

for all $q \in (0, 1/2)$ and all weights $w \in \mathcal{I}_n$ such that $w \neq \underline{w}$ and w is not a permutation of $\overline{w}$.

Theorem 4.4 *Let $q \in (0, 1/2)$, $r \in (0, 2]$, and let $X_1, ..., X_n$ be i.i.d. risks such that $X_i \sim \underline{\mathcal{CS}}(r)$, $i = 1, ..., n$. Then* *(i)* $VaR_q(Z_v) > VaR_q(Z_w)$ *if* $(v_1^r, ..., v_n^r) \prec (w_1^r, ..., w_n^r)$ *and* $(v_1^r, ..., v_n^r)$ *is not a permutation of* $(w_1^r, ..., w_n^r)$ *(that is, the function* $\psi(w, q) = VaR_q(Z_w)$, $w \in \mathbf{R}_+^n$ *is strictly Schur-concave in* $(w_1^r, ..., w_n^r)$*).*

(ii) The following sharp bounds hold :

$$(\sum_{i=1}^n w_i^r)^{1/r} VaR_q(Z_{\overline{w}}) < VaR_q(Z_w) < n^{1-1/r} (\sum_{i=1}^n w_i^r)^{1/r} VaR_q(Z_{\underline{w}})$$

for all $q \in (0, 1/2)$ *and all weights* $w \in \mathcal{I}_n$ *such that* $w \neq \underline{w}$ *and* w *is not a permutation of* $\overline{w}$.

Remark 4.1 *It is well-known that if r.v.'s* X *and* Y *are such that* $P(X > x) \leq P(Y > x)$ *for all* $x \in \mathbf{R}$, *then* $EU(X) \leq EU(Y)$ *for all increasing functions* $U : \mathbf{R} \to \mathbf{R}$ *for which the expectations exist (see Shaked and Shanthikumar, 1994, pp. 3-4). This fact, together with Theorems 4.1-4.4 imply corresponding results concerning majorization properties of expectations of (utility or payoff) functions of linear combinations of heavy-tailed r.v.'s. For instance, we get that if* $U : \mathbf{R}_+ \to \mathbf{R}$ *is an increasing function, then, assuming existence of the expectations, the function* $\varphi(w) = EU(|\sum_{i=1}^n w_i X_i|)$, $w \in \mathbf{R}_+^n$, *is Schur-convex in* $(w_1^r, ..., w_n^r)$ *under the assumptions of Theorem 4.3 and is Schur-concave in* $(w_1^r, ..., w_n^r)$ *under the assumptions of Theorem 4.4. In particular,*

$$EU\big(\big|n^{1-1/r}\big(\sum_{i=1}^n w_i^r\big)^{1/r} Z_{\underline{w}}\big|\big) \leq EU(|Z_w|) \leq EU\big(\big|\big(\sum_{i=1}^n w_i^r\big)^{1/r} Z_{\overline{w}}\big|\big)$$

for all portfolios of risks satisfying Theorem 4.3 and

$$EU\big(\big|\big(\sum_{i=1}^n w_i^r\big)^{1/r} Z_{\overline{w}}\big|\big) \leq EU(|Z_w|) \leq EU\big(\big|n^{1-1/r}\big(\sum_{i=1}^n w_i^r\big)^{1/r} Z_{\underline{w}}\big|\big)$$

for all portfolios of risks satisfying Theorem 4.4. We also get that the function $\varphi(w)$, $w \in \mathbf{R}_+^n$ *is Schur-concave in* $(w_1^2, ..., w_n^2)$ *if* $X_i \sim S_\alpha(\sigma, \beta, 0)$, $i = 1, ..., n$, *for some* $\sigma > 0$, $\beta \in [-1, 1]$ *and* $\alpha \in (0, 2)$, *or* $X_i \sim \underline{\mathcal{CS}}(2)$. *The above results extend and complement those in Efron (1969) and Eaton (1970) (see also Marshall and Olkin, 1979, pp. 361-365) who studied classes of functions* $U : \mathbf{R} \to \mathbf{R}$ *and r.v.'s* $X_1, ..., X_n$ *for which Schur-concavity of* $\varphi(w)$, $w \in \mathbf{R}_+^n$, *in* $(w_1^2, ..., w_n^2)$ *holds. Further, we obtain that* $\varphi(w)$ *is Schur-convex in* $w \in \mathbf{R}_+^n$ *under the assumptions of Theorem 4.1 and is Schur-concave in* $a \in \mathbf{R}_+^n$ *under the assumptions of Theorem 4.2. It is important to note here that in the case of increasing convex functions* $U : \mathbf{R}_+ \to \mathbf{R}$ *and r.v.'s* $X_1, ..., X_n$ *satisfying the assumptions of Theorem 4.2, the expectations* $EU(|\sum_{i=1}^n w_i X_i|)$ *are infinite for all* $w \in \mathbf{R}_+^n$.[11] *Therefore, the last result does not contradict the well-known fact that (see Marshall and Olkin, 1979, p. 361) the function* $Ef(\sum_{i=1}^n w_i X_i)$ *is Schur-convex in* $(w_1, ..., w_n) \in \mathbf{R}$ *for all i.i.d. r.v.'s* $X_1, ..., X_n$ *and convex functions* $f : \mathbf{R} \to \mathbf{R}$ *as it might seem on the first sight.*

Remark 4.2 *If r.v.'s* $X_1, ..., X_n$ *have a symmetric Cauchy distribution* $S_1(\sigma, 0, 0)$ *which is, as discussed in Subsection 2, exactly at the dividing boundary between the class* $\underline{\mathcal{CS}}(1)$ *in Theorem 4.1 and the class* $\overline{\mathcal{CSLC}}$ *in Theorem 4.2, then the value at risk* $VaR_q(Z_w)$ *depends only on* $\sum_{i=1}^n w_i$ *and* α *and is thus the same for all portfolio of risks* X_i, $i = 1, ..., n$. *Consequently, in such a case, diversification of a portfolio has no effect on riskiness of its return. Similarly, the value at risk function* $\psi(w, q) = VaR_q(Z_w)$, $w \in \mathbf{R}_+^n$ *in Theorems 4.3 and 4.4 is both Schur-concave and Schur-convex in* $(w_1^r, ..., w_n^r)$ *if the risks* $X_1, ..., X_n$ *in the theorems have a symmetric stable distribution*

[11] Since the function $(f(x) - f(0))/x$ is increasing in $x > 0$ by, e.g., Marshall and Olkin (1979), p. 453.

$S_r(\sigma,0,0)$ with the index of stability $\alpha = r$ which is at the dividing boundary between the classes $\overline{\mathcal{CS}}(r)$ and $\underline{\mathcal{CS}}(r)$. From the proof of Theorems 4.1-4.4, it follows that Theorems 4.1 and 4.2 continue to hold for convolutions of distributions from the classes $\overline{\mathcal{CSLC}}$ and $\underline{\mathcal{CS}}(1)$ with symmetric Cauchy distributions $S_1(\sigma,0,0)$. Similarly, Theorem 4.3 and 4.4 continue to hold for convolutions of distributions from the classes $\overline{\mathcal{CS}}(r)$ and $\underline{\mathcal{CS}}(r)$ with symmetric stable distributions $S_r(\sigma,0,0)$.

Remark 4.3 *In complete similarity to the proof of Theorems 4.3 and 4.4, it is not difficult to obtain their analogues for i.i.d. risks $X_1, ..., X_n$ with skewed stable distributions $X_i \sim S_\alpha(\sigma,\beta,0)$, $i = 1,...,n$.*

Remark 4.4 *Theorems 4.1-4.4 imply corresponding results on majorization properties of the tail probabilities $\xi(a,x) = P(\sum_{i=1}^n w_i X_i > x)$, $x > 0$, of linear combinations of r.v.'s $X_1,...,X_n$. These implications provide substantial generalizations of the results in the seminal work of Proschan (1965) who showed that the tail probabilities $\xi(a,x)$ are Schur-convex in $a = (a_1,...,a_n) \in \mathbf{R}_+^n$ for all $x > 0$ for r.v.'s $X_i \sim \mathcal{LC}$, $i = 1,...,n$, with symmetric log-concave distributions.[12] [13] Proschan's (1965) results and their extensions have been applied to the analysis of many problems in statistics, econometrics, economic theory, mathematical evolutionary theory and other fields. For instance, Eaton (1988) used generalizations of the results to obtain concentration inequalities for Gauss-Markov estimators. Karlin (1984, 1992) applied them in the study of environmental sex determination models. Jovanovic and Rob (1987) used majorization properties of log-concavely distributed r.v.'s derived by Proschan (1965) in the analysis of the model of demand-driven innovation and spatial competition over time. Fang and Norman (2003) applied them in the study of optimal bundling strategies for a multiproduct monopolist. Several authors (see, e.g., Proschan, 1965, Tong, 1994, and Jensen, 1997) discussed implications of the majorization results for log-concave distributions and their extensions in the study of monotone consistency of estimators in statistics and econometrics. One should note here that applicability of these majorization results and their analogs for other classes of distributions to portfolio value at risk theory has not yet been recognized in the literature even in the case of i.i.d. log-concavely distributed risks.*

A number of papers in probability and statistics have focused on extension of Proschan's results (see, among others, Chan, Park and Proschan, 1989, the review in Tong, 1994, Jensen, 1997, and Ma, 1998). One should emphasize, however, that in all the studies that dealt with generalizations of the results, the majorization properties of the tail probabilities were of the same type as in Proschan (1965). Namely, the results gave extensions of Proschan's results concerning Schur−convexity of the tail probabilities $\xi(a,x)$, $x > 0$, to classes of r.v.'s more general than those considered in Proschan (1965). Analogues of Theorems 4.2 and 4.4 for the tail probabilities $\xi(a,x)$, on the other hand, provide general results concerning Schur−concavity of $\xi(a,x)$, $x > 0$, for certain wide classes of r.v.'s. According to these results, the class of distributions for which Schur-convexity of the tail probabilities $\xi(a,x)$ is replaced by their Schur-concavity is precisely the class of distributions with extremely thick-tailed densities.[14]

[12] Proschan (1965) notes that similar majorization orderings also hold for (two-fold) convolutions of log-concave distributions with symmetric Cauchy distributions and shows that peakedness comparisons implied by them are reversed for $n = 2^k$, vectors $a = (1/n, 1/n, ..., 1/n) \in \mathbf{R}^n$ with identical components and certain transforms of symmetric Cauchy r.v.'s.

[13] The main results in Proschan (1965) are reviewed in Section 12.J in Marshall and Olkin (1979). Proschan's (1979) work is also presented, in a rearranged form, in Section 11 of Chapter 7 in Karlin (1968). Peakedness results in Proschan (1965) and Karlin (1968) are formulated for "PF2 densities," which is the same as "log-concave densities."

[14] One should note that the proof in Proschan (1965) can be reproduced word to word with respective changes of signs of inequalities under the "assumptions" that $X_1,...,X_n$ are i.i.d. symmetric log-convexily distributed r.v.'s. However, as it is easy to see, the later objects do not exist, namely, there does not exist a symmetric r.v.'s with a log-convex density (see also An, 1998). Therefore, this approach to obtaining counterparts of the results in Proschan (1965) for Schur-concavity of $\xi(a,x)$, $x > 0$, is hopeless.

5 Main results: (Non-)Coherency of the value at risk under thick-tailedness

The present section contains the main results of the paper on coherency properties of the VaR in the world of not extremely heavy-tailed risks and on its non-coherency for risks with extremely thick-tailed distributions.

Let $\mathcal{X}$ be a certain linear space of r.v.'s X defined on a probability space $(\Omega, \Im, P)$. We assume that $\mathcal{X}$ contains all degenerate r.v.'s $X \equiv a \in \mathbf{R}$. According to the definition in Artzner et. al. (1999) (see also Embrechts et. al., 2002, and Frittelli and Gianin, 2002), a functional $\mathcal{R} : \mathcal{X} \to R$ is said to be a *coherent* measure of risk if it satisfies the following axioms:

A1. (Monotonicity) $\mathcal{R}(X) \geq \mathcal{R}(Y)$ for all $X, Y \in \mathcal{X}$ such that $Y \leq X$ (a.s.), that is, $P(X \leq Y) = 1$.

A2. (Translation invariance) $\mathcal{R}(X + a) = \mathcal{R}(X) + a$ for all $X \in \mathcal{X}$ and any $a \in \mathbf{R}$.

A3. (Positive homogeneity) $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for all $X \in \mathcal{X}$ and any $\lambda \geq 0$.

A4. (Subadditivity) $\mathcal{R}(X + Y) \leq \mathcal{R}(X) + \mathcal{R}(Y)$ for all $X, Y \in \mathcal{X}$.

In some papers (see, e.g., Frittelli and Gianin, 2002, and Fölmer and Schied, 2002), the axioms A3 and A4 were replaced by the following weaker axiom of convexity:

A5. (Convexity) $\mathcal{R}(\lambda X + (1 - \lambda)Y) \leq \lambda \mathcal{R}(X) + (1 - \lambda)\mathcal{R}(Y)$ for all $X, Y \in \mathcal{X}$ and any $\lambda \in [0, 1]$

(clearly, A5 follows from A3 and A4). The above axioms are natural conditions to be imposed on measures of risk in the setting where positive values of r.v.'s $X \in \mathcal{X}$ represent losses of a risk holder.[15] For instance, subadditivity property is important, among others, from the regulatory point of view because if a firm were forced to meet the requirement of extra capital which is not subadditive, it might be motivated to break up into several separately incorporated affiliates (see the discussion in Artzner et. al., 1999). In addition to that, the properties A1-A5 are important because, as follows from Huber (1981, Ch. 10) (see also Artzner et. al., 1999), in the case of a finite Ω, a risk measure $\mathcal{R}$ is coherent (that is, it satisfies A1-A4) if and only if it is representable as $\mathcal{R}(X) = \sup_{Q \in \mathcal{P}} E_Q(X)$, where $\mathcal{P}$ is some set of probability measures on Ω and, for $Q \in \mathcal{P}$, E_Q denotes the expectation with respect to Q. In other words, the risk measure $\mathcal{R}$ is the worst result of computing the expected loss $E_Q(X)$ over a set $\mathcal{P}$ of "generalized scenarios" (probability measures) Q. A similar representation holds as well in the case of an arbitrary Ω and the space $\mathcal{X} = L^\infty(\Omega, \Im, P)$ of bounded r.v.'s (see Fölmer and Schied, 2002); moreover, as discussed in Frittelli and Gianin (2002), by duality theory, the convexity axiom A5 alone implies analogues of such characterizations for an arbitrary Ω and the space $\mathcal{X} = L_p(\Omega, \Im, P)$, $p \geq 1$, of r.v.'s X with a finite $p-$th moment $E|X|^p < \infty$.

It is easy to verify that the value at risk $VaR_q(X)$ satisfies the axioms of monotonicity, positive homogeneity and translation invariance A1, A3 and A4. However, as follows from the counterexamples constructed by Artzner et. al. (1999) and Embrechts et. al. (2002), in general, it fails to satisfy the subadditivity and convexity properties A2 and A5, in particular, for certain Pareto distributions (Examples 6 and 7 in Embrechts et. al., 2002).

[15] This interpretation of losses follows that in Embrechts et. al. (2002) and is in contrast to Artzner et. al. (1999) who interpret *negative* values of risks in $\mathcal{X}$ as losses.

According to the following theorem, the value at risk satisfies subadditivity and convexity axioms A4 and A5 and is, thus, a coherent measure of risk in the world of long-tailed risks from the class $\overline{\mathcal{CSLC}}$:

Theorem 5.1 *Let X_1 and X_2 be i.i.d. risks such that $X_i \sim \overline{\mathcal{CSLC}}$, $i = 1, 2$. For all $q \in (0, 1/2)$ and any $\lambda \in (0, 1)$, one has $VaR_q(X_1 + X_2) < VaR_q(X_1) + VaR_q(X_2)$ and $VaR_q(\lambda X_1 + (1-\lambda)X_2) < \lambda VaR_q(X_1) + (1-\lambda)VaR_q(X_2)$. That is, subadditivity and convexity axioms A4 and A5 are satisfied for VaR and it is a coherent measure of risk for the class $\overline{\mathcal{CSLC}}$.*

Theorem 5.2 shows that axioms A2 and A5 are *always* violated for risks with extremely heavy-tailed distributions. Thus, the value at risk is not a coherent risk measure in the world of extremely long-tailed distributions:

Theorem 5.2 *Let X_1 and X_2 be i.i.d. risks such that $X_i \sim \underline{\mathcal{CS}}(1)$, $i = 1, 2$. For all $\alpha \in (0, 1/2)$ and any $\lambda \in (0, 1)$, one has $VaR_q(X_1) + VaR_q(X_2) < VaR_q(X_1 + X_2)$ and $\lambda VaR_q(X_1) + (1-\lambda)VaR_q(X_2) < VaR_q(\lambda X_1 + (1-\lambda)X_2)$. That is, subadditivity and convexity axioms A4 and A5 are violated for VaR and it is not a coherent measure of risk for the class $\underline{\mathcal{CS}}(1)$.*

6 Extensions and conclusion

6.1 Generalizations to dependence and non-identical distributions

As indicated in Subsection 1.3 in the introduction, the results obtained in this paper continue to hold for wide classes of dependent and non-identically distributed r.v.'s. More precisely, the results continue to hold for convolutions of r.v.'s with joint $\alpha-$symmetric and spherical distributions and their non-identically distributed versions as well as for a wide class of models with common shocks.

According to the definition introduced by Cambanis, Keener and Simons (1983), an $n-$dimensional distribution is called $\alpha-$symmetric if its characteristic function (c.f.) can be written as $\phi((\sum_{i=1}^{n} |t_i|^{\alpha})^{1/\alpha})$, where $\phi : \mathbf{R}_+ \to \mathbf{R}$ is a continuous function and $\alpha > 0$. The number α is called the index and the function ϕ is called the c.f. generator of the $\alpha-$symmetric distribution. The class of $\alpha-$symmetric distributions is very broad and contains, in particular, spherical distributions corresponding to the case $\alpha = 2$ (see Fang, Kotz and Ng, 1990, p. 184). Spherical distributions, in turn, include such important examples as Kotz type, multinormal, multivariate t and multivariate stable laws (Fang et. al., 1990, Ch. 3). Furthermore, for any $0 < \alpha \leq 2$, the class of $\alpha-$symmetric distributions includes distributions of risks $X_1, ..., X_n$ that have the representation

$$(X_1, ..., X_n) = (ZY_1, ..., ZY_n) \tag{5}$$

where $Y_i \sim S_{\alpha}(\sigma, 0, 0)$ are i.i.d. symmetric stable r.v.'s with $\sigma > 0$ and the index of stability α and $Z \geq 0$ is a nonnegative r.v. independent of $Y_i's$ (see Fang et. al., 1990, p. 197). Models (5) and their convolutions belong to the class of models with common shocks Z, such as macroeconomic or political ones, that affect all risks Y_i.

It is important to emphasize here that the necessity in the study of effects of common shocks arises in many areas of economics and finance (see Andrews, 2003). The extensions of the results in this paper to such models provides, in fact, a new approach to the analysis of robustness of portfolio value at risk theory and many other models in economics, finance and risk management to both heavy-tailedness and to common shocks.

In addition, one should indicate here that the extensions of the results to $\alpha-$symmetric and, in particular, spherical distributions cover many thick-tailed models with finite variances and finite higher moments. For instance, multivariate $t-$distributions that belong to the class of spherical distributions, provide one of now well-established approaches to modelling heavy-tailedness phenomena with moments up to some order (see Praetz, 1972, Blattberg and Gonedes, 1974, and Glasserman et. al., 2002). The following theorems provide precise formulations of the extensions of the results in Sections 4 and 5 to the dependent case. According to the theorems, all the results presented in those sections for convolutions of i.i.d. stable distributions with indices of stability α belonging to a certain range (and convolutions of those with log-concave distributions in the case of the class $\overline{\mathcal{CSLC}}$) continue to hold for convolutions of $\alpha-$symmetric distributions and models with common shocks (5) with parameters α in the same range.

Let Φ denote the class of c.f. generators ϕ such that $\phi(0) = 1$, $\lim_{t\to\infty} \phi(t) = 0$, and the function $\phi'(t)$ is concave.

Theorem 6.1 *Theorems 4.1 and 5.1 continue to hold if any of the following is satisfied:*

the random vector $(X_1, ..., X_n)$ entering their assumptions is a sum of i.i.d. random vectors $(Y_{1j}, ..., Y_{nj})$, $j = 1, ..., k$, where $(Y_{1j}, ..., Y_{nj})$ has an absolutely continuous $\alpha-$symmetric distribution with the c.f. generator $\phi_j \in \Phi$ and the index $\alpha_j \in (1, 2]$. In particular, the results hold when the vector of r.v.'s entering their assumptions is a sum of i.i.d. random vectors $(Y_{1j}, ..., Y_{nj})$, $j = 1, ..., k$, that have absolutely continuous spherical distributions with c.f. generators $\phi_j \in \Phi$ (the case $\alpha_j = 2$ for all j).

the vector of r.v.'s $(X_1, ..., X_n)$ entering the assumptions of the results is a sum of i.i.d. random vectors $(Z_jY_{1j}, ..., Z_jY_{nj})$, $j = 1, ..., k$, where $Y_{ij} \sim S_{\alpha_j}(\sigma_j, 0, 0)$, $i = 1, ..., n$, $j = 1, ..., k$, with $\sigma_j > 0$ and $\alpha_j \in (1, 2]$ and Z_j are absolutely continuous positive r.v.'s independent of Y_{ij}.

Theorem 6.2 *Theorems 4.2 and 5.2 continue to hold if any of the following is satisfied:*

the vector of r.v.'s $(X_1, ..., X_n)$ entering their assumptions is a sum of i.i.d. random vectors $(Y_{1j}, ..., Y_{nj})$, $j = 1, ..., k$, where $(Y_{1j}, ..., Y_{nj})$ has an absolutely continuous $\alpha-$symmetric distribution with the c.f. generator $\phi_j \in \Phi$ and the index $\alpha_j \in (0, 1)$;

the vector of r.v.'s entering the assumptions of the results is a sum of i.i.d. random vectors $(Z_jY_{1j}, ..., Z_jY_{nj})$, $j = 1, ..., k$, where $Y_{ij} \sim S_{\alpha_j}(\sigma_j, 0, 0)$, $i = 1, ..., n$, $j = 1, ..., k$, with $\sigma_j > 0$ and $\alpha_j \in (0, 1)$ and Z_j are positive absolutely continuous r.v.'s independent of Y_{ij}.

Theorem 6.3 *Theorem 4.3 continues to hold under any of the following two assumptions:*

the random vector $(X_1, ..., X_n)$ is a sum of i.i.d. random vectors $(Y_{1j}, ..., Y_{nj})$, $j = 1, ..., k$, where $(Y_{1j}, ..., Y_{nj})$ has an absolutely continuous $\alpha-$symmetric distribution with the c.f. generator $\phi_j \in \Phi$ and the index $\alpha_j \in (r, 2]$;

the random vector $(X_1, ..., X_n)$ is a sum of i.i.d. random vectors $(Z_jY_{1j}, ..., Z_jY_{nj})$, $j = 1, ..., k$, where $Y_{ij} \sim$

$S_{\alpha_j}(\sigma_j, 0, 0),\ i = 1, ..., n,\ j = 1, ..., k,$ *with* $\sigma_j > 0$ *and* $\alpha_j \in (r, 2]$ *and* Z_j *are positive absolutely continuous r.v.'s independent of* Y_{ij}.

Theorem 6.4 *Theorem 4.4 continues to hold if any of the following assumptions is satisfied:*

the random vector $(X_1, ..., X_n)$ *is a sum of i.i.d. random vectors* $(Y_{1j}, ..., Y_{nj}),\ j = 1, ..., k,$ *where* $(Y_{1j}, ..., Y_{nj})$ *has an absolutely continuous* $\alpha-$*symmetric distribution with the c.f. generator* $\phi_j \in \Phi$ *and the index* $\alpha_j \in (0, r)$;

the random vector $(X_1, ..., X_n)$ *is a sum of i.i.d. random vectors* $(Z_jY_{1j}, ..., Z_jY_{nj}),\ j = 1, ..., k,$ *where* $Y_{ij} \sim S_{\alpha_j}(\sigma_j, 0, 0),\ i = 1, ..., n,\ j = 1, ..., k,$ *with* $\sigma_j > 0$ *and* $\alpha_j \in (0, r)$ *and* Z_j *are positive absolutely continuous r.v.'s independent of* Y_{ij}.

As for generalizations of the results in the paper to the case of non-identical distributions, the following conclusions hold.

Let $\sigma_1, ..., \sigma_n \geq 0$ be some scale parameters and let $X_i \sim S_\alpha(\sigma_i, \beta, 0)$, $\alpha \in (0, 2]$, be independent non-identically distributed stable risks. Further, let $Z_{\tilde{w}} = \sum_{i=1}^n w_{[i]} X_i$ be the return on the portfolio with weights $\tilde{w} = (w_{[1]}, ..., w_{[n]})$, where, as in Section 3, $w_{[1]} \geq ... \geq w_{[n]}$ denote the components of the vector $w = (w_1, ..., w_n) \in \mathbf{R}_+^n$ in decreasing order.[16] Similar to the proof of Theorems 4.1, 4.3 and 5.1, one can show that the theorems continue to hold (in the same range of parameters r and α) for the returns $Z_{\tilde{w}}$ if $\sigma_1 \geq ... \geq \sigma_n \geq 0$. Similarly, Theorems 4.2, 4.4 and 5.2 continue to hold for the returns $Z_{\tilde{w}}$ if $\sigma_n \geq ... \geq \sigma_1 \geq 0$.[17] Using conditioning arguments, one gets that these extensions also hold in the case of random scale parameters σ_i.

7 Concluding remarks

The present paper developed a unified approach to the analysis of portfolio value at risk under heavy-tailedness using new majorization theory for linear combinations of thick-tailed r.v.'s. Our main results show that the stylized facts on portfolio diversification hold and the value at risk exhibits the important property of coherency in the world of not extremely heavy-tailed risks. However, according to the results obtained in the paper, the stylized fact that portfolio diversification is always preferable is reversed and value at risk coherency is violated in the case of risks with sufficiently long-tailed distributions.

In other words, the stylized facts on portfolio diversification and the property of value at risk coherency are robust to the assumptions of heavy-tailedness of risks if the distributions entering these assumptions are not extremely thick-tailed. These facts and the VaR coherency property are not robust to the assumptions involving extremely heavy-tailed risks. Furthermore, as follows from the extensions discussed in the paper, in addition to heavy-tailedness,

[16] A certain ordering in the components of the vector of weights w is necessary for the extensions of the majorization results in this paper to the case of non-identically distributed r.v.'s X_i since Schur-convexity and Schur-concavity of a function $f(a)$ in a imply its symmetry in the components of a.

[17] These results for $Z_{\tilde{w}}$ can be established in the same way as Theorems 4.1-5.2 using the fact that, by Theorem 3.A.4 in Marshall and Olkin (1979), the function $\chi(c_1, ..., c_n) = \sum_{i=1}^n \sigma_i^\alpha c_{[i]}^\alpha$ is strictly Schur-convex in $(c_1, ..., c_n) \in \mathbf{R}_+^n$ if $\alpha > 1$ and $\sigma_1 \geq ... \geq \sigma_n \geq 0$ and is strictly Schur-concave in $(c_1, ..., c_n) \in \mathbf{R}_+^n$ if $\alpha < 1$ and $\sigma_n \geq ... \geq \sigma_1 \geq 0$.

the paper accomplishes the unification of the analysis of robustness of value at risk models to such important distributional phenomena as dependence, skewness and the case of non-identical marginals.

8 Proofs.

In the proofs of the results in Section 4 below, we provide the complete argument for the results on diversification that provide reversals of the stylized facts on portfolio diversification, namely for Theorem 4.2 and Theorem 4.4. The proof of Theorem 4.3 that gives the results on Schur-convexity of the value at risk function $\psi(w, q) = VaR_q(Z_w)$ follows the same lines as that of Theorem 4.4, with respective changes in the signs of inequalities. We also provide the complete proof of Theorem 4.1 since it is not implied by Theorem 4.3 alone, but needs to combine the results in that theorem with those for the tail probabilities of log-concavely distributed r.v.'s obtained in Proschan (1965) (see Remark 4.4).

Proof of Theorems 4.3 and 4.4. Let $r, \alpha \in (0, 2]$, $\sigma > 0$, and let $v = (v_1, ..., v_n) \in \mathbf{R}_+^n$ and $w = (w_1, ..., w_n) \in \mathbf{R}_+^n$ be two vectors of portfolio weights such that $(v_1^r, ..., v_n^r) \prec (w_1^r, ..., w_n^r)$ and $(v_1^r, ..., v_n^r)$ is not a permutation of $(w_1^r, ..., w_n^r)$ (clearly, $\sum_{i=1}^n v_i \neq 0$ and $\sum_{i=1}^n w_i \neq 0$). Let $X_1, ..., X_n$ be i.i.d. risks such that $X_i \sim S_\alpha(\sigma, 0, 0)$, $i = 1, ..., n$. It is not difficult to see that if $c = (c_1, ..., c_n) \in \mathbf{R}_+^n$, $\sum_{i=1}^n c_i \neq 0$, then $Z_c / \left(\sum_{i=1}^n c_i^\alpha\right)^{1/\alpha} = \sum_{i=1}^n c_i X_i / \left(\sum_{i=1}^n c_i^\alpha\right)^{1/\alpha} \sim S_\alpha(\sigma, 0, 0)$. Consequently, by Axiom (A3) in Section 5 which is satisfied for the value at risk, for all $q \in (0, 1/2)$ we have

$$VaR_q(Z_c) = VaR_q(X_1)\Big(\sum_{i=1}^n c_i^\alpha\Big)^{1/\alpha}. \tag{4}$$

According to Proposition 3.C.1.a in Marshall and Olkin (1979), the function $\phi(c_1, ..., c_n) = \sum_{i=1}^n c_i^\alpha$ is strictly Schur-convex in $(c_1, ..., c_n) \in \mathbf{R}_+^n$ if $\alpha > 1$ and is strictly Schur-concave in $(c_1, ..., c_n) \in \mathbf{R}_+^n$ if $\alpha < 1$. Therefore, we have $\sum_{i=1}^n v_i^\alpha = \sum_{i=1}^n (v_i^r)^{\alpha/r} < \sum_{i=1}^n (w_i^r)^{\alpha/r} = \sum_{i=1}^n w_i^\alpha$, if $\alpha/r > 1$ and $\sum_{i=1}^n w_i^\alpha = \sum_{i=1}^n (w_i^r)^{\alpha/r} < \sum_{i=1}^n (v_i^r)^{\alpha/r} = \sum_{i=1}^n v_i^\alpha$, if $\alpha/r < 1$. This, together with (4) implies that, for all $q \in (0, 1/2)$,

$$VaR_q(Z_v) < VaR_q(Z_w) \tag{5}$$

if $\alpha > r$, and

$$VaR_q(Z_v) > VaR_q(Z_w) \tag{6}$$

if $\alpha < r$. This completes the proof of parts (i) of the theorems in the case of i.i.d. stable risks $X_i \sim S_\alpha(\sigma, 0, 0)$, $i = 1, ..., n$.

Suppose now that $X_1, ..., X_n$ are i.i.d. risks such that $X_i \sim \underline{\mathcal{CS}}(r)$, $i = 1, ..., n$. By definition of the class $\underline{\mathcal{CS}}(r)$, there exist independent r.v.'s Y_{ij}, $i = 1, ..., n$, $j = 1, ..., k$, such that $Y_{ij} \sim S_{\alpha_j}(\sigma_j, 0, 0)$, $\alpha_j \in (0, r)$, $\sigma_j > 0$, $i = 1, ..., n$, $j = 1, ..., k$, and $X_i = \sum_{j=1}^k Y_{ij}$, $i = 1, ..., n$. By (6), for all $q \in (0, 1/2)$ and all $j = 1, ..., k$,

$$VaR_q\Big(\sum_{i=1}^n v_i Y_{ij}\Big) > VaR_q\Big(\sum_{i=1}^n w_i Y_{ij}\Big). \tag{7}$$

The r.v.'s Y_{ij}, $i = 1, ..., n$, $j = 1, ..., k$, are symmetric and unimodal by Theorem 2.7.6 in Zolotarev (1986, p. 134). Therefore, from Theorem 1.6 in Dharmadhikari and Joag-Dev (1988, p. 13) it follows that the r.v.'s $\sum_{i=1}^n v_i Y_{ij}$, $j = 1, ..., k$, and $\sum_{i=1}^n w_i Y_{ij}$, $j = 1, ..., k$, are symmetric and unimodal as well. From Lemma in Birnbaum (1948) and its proof it follows that if X_1, X_2 and Y_1, Y_2 are independent absolutely continuous symmetric unimodal r.v.'s such that, for $i = 1, 2$, and all $q \in (0, 1/2)$, $VaR_q(X_i) < VaR_q(Y_i)$, then $VaR_q(X_1 + X_2) < VaR_q(Y_1 + Y_2)$ for all $q \in (0, 1/2)$. This, together with (7) and symmetry and unimodality of $\sum_{i=1}^n v_i Y_{ij}$ and $\sum_{i=1}^n w_i Y_{ij}$, $j = 1, ..., k$, imply, by induction on k (see also Theorem 1 in Birnbaum, 1948, and Theorem 2.C.3 in Dharmadhikari and Joag-Dev, 1988), that $\psi(v, q) = VaR_q(Z_v) = VaR_q\big(\sum_{j=1}^k \sum_{i=1}^n v_i Y_{ij}\big) > VaR_q\big(\sum_{j=1}^k \sum_{i=1}^n w_i Y_{ij}\big) = VaR_q(Z_w) = \psi(w, q)$ for all $q \in (0, 1/2)$. Therefore, part (i) of Theorem 4.3 holds. Part (i) of Theorem 4.4 might be proven in a completely similar way.

The bounds in parts (ii) of Theorems 4.3 and 4.4 follow from their parts (i) and the fact that, by majorization comparisons (3), $\big(\sum_{i=1}^n w_i^r/n, ..., \sum_{i=1}^n w_i^r/n\big) \prec (w_1^r, ..., w_n^r) \prec \big(\sum_{i=1}^n w_i^r, 0, ..., 0\big)$ for all portfolio weights $w \in \mathbf{R}_+^n$ and all $r \in (0, 2]$. Sharpness of the bounds in the theorems follows from the fact that, as it is not difficult to see, the bounds become equalities in the limit as $\alpha \to r$ for symmetric stable risks $X_i \sim S_\alpha(\sigma, 0, 0)$, $i = 1, ..., n$. ■

Proofs of Theorems 4.1 and 4.2 and Theorems 5.1 and 5.2. Theorem 4.1 for the case of i.i.d. stable risks $X_i \sim S_\alpha(\sigma, 0, 0)$, $i = 1, ..., n$, and Theorem 4.2 for distributions from the class $\underline{\mathcal{CS}}(1)$ are immediate consequences of Theorems 4.3 and 4.4 with $r = 1$. Let us prove Theorem 4.1 for the case of the class $\overline{\mathcal{CSLC}}$. Let the vectors of portfolio weights $v = (v_1, ..., v_n) \in \mathbf{R}_+^n$ and $w = (w_1, ..., w_n) \in \mathbf{R}_+^n$ be such that $v \prec w$ and v is not a permutation of w. Further, let $X_1, ..., X_n$ be i.i.d. risks such that $X_i \sim \overline{\mathcal{CSLC}}$, $i = 1, ..., n$. By definition, $X_i = \gamma Y_{i0} + \sum_{j=1}^k Y_{ij}$, $i = 1, ..., n$, where $\gamma \in \{0, 1\}$, $k \geq 0$ and $(Y_{1j}, ..., Y_{nj})$, $j = 0, 1, ..., k$, are independent vectors with i.i.d. components such that $Y_{i0} \sim \mathcal{LC}$, $i = 1, ..., n$, and $Y_{ij} \sim S_{\alpha_j}(\sigma_j, 0, 0)$, $\alpha_j \in (1, 2]$, $\sigma_j > 0$, $i = 1, ..., n$, $j = 1, ..., k$. From (5) and the results in Proschan (1965) for tail probabilities of log-concavely distributed r.v.'s (see Remark 4.4) it follows that, for all $q \in (0, 1/2)$ and all $j = 1, ..., k$, $VaR_q\big(\sum_{i=1}^n v_i Y_{ij}\big) < VaR_q\big(\sum_{i=1}^n w_i Y_{ij}\big)$. Furthermore, from Theorem 2.7.6 in Zolotarev (1986, p. 134) and Theorems 1.6 and 1.10 in Dharmadhikari and Joag-Dev (1988, pp. 13 and 20) by induction it follows that the r.v.'s $\sum_{i=1}^n v_i Y_{ij}$ and $\sum_{i=1}^n w_i Y_{ij}$, $j = 0, 1, ..., k$, are symmetric and unimodal. Similar to the proof of Theorems 4.3 and 4.4, by Lemma in Birnbaum (1948) and its proof and induction, this implies that

$$VaR_q(Z_v) = VaR_q\big(\sum_{i=1}^n v_i X_i\big) = VaR_q\big(\gamma \sum_{i=1}^n v_i Y_{i0} + \sum_{j=1}^k \sum_{i=1}^n v_i Y_{ij}\big) <$$

$$VaR_q\big(\sum_{i=1}^n w_i X_i\big) = VaR_q\big(\gamma \sum_{i=1}^n w_i Y_{i0} + \sum_{j=0}^k \sum_{i=1}^n w_i Y_{ij}\big).$$

This completes the proof of part (i) of Theorem 4.2. Part (ii) of Theorem 4.2 follows from its part (i) and majorization comparisons (3). Theorems 5.1 and 5.2 immediately follow from Theorems 4.1 and 4.2. ■

Proof of Theorems 6.1-6.4. The proof of the extensions of Theorems 4.1-5.2 to the dependent case follows the same lines as the proof of the above theorems since the following properties hold:

$\sum_{i=1}^n c_i X_i / \big(\sum_{i=1}^n c_i^\alpha\big)^{1/\alpha}$ has the same distribution as that of X_1 if $(X_1, ..., X_n)$ has an $\alpha-$symmetric distribution (see, e.g., Fang, Kotz and Ng, 1990, Ch. 7);

The r.v.'s $\sum_{i=1}^{n} v_i Y_{ij}$, $j = 1, ..., k$, and $\sum_{i=1}^{n} w_i Y_{ij}$, $j = 1, ..., k$, are symmetric and unimodal if $(Y_{1j}, ..., Y_{nj})$ has an $\alpha-$symmetric distribution with the c.f. generator $\phi_j \in \Phi$ (this easily follows from a result due to R. Askey, see Theorem 4.1 in Gneiting, 1998);

The r.v.'s $Z_j \sum_{i=1}^{n} v_i Y_{ij}$, $j = 1, ..., k$, and $Z_j \sum_{i=1}^{n} w_i Y_{ij}$, $j = 1, ..., k$, are symmetric and unimodal if $Y_{ij} \sim S_{\alpha_j}(\sigma_j, 0, 0)$, $i = 1, ..., n$, $j = 1, ..., k$, and Z_j are positive r.v.'s independent of Y_{ij} (this follows by symmetry and unimodality of $\sum_{i=1}^{n} v_i Y_{ij}$ and $\sum_{i=1}^{n} w_i Y_{ij}$ implied by Theorem 2.7.6 in Zolotarev, 1986, p. 134, and Theorem 1.6 in Dharmadhikari and Joag-Dev, 1988, p. 13, the definition of unimodality and conditioning arguments). ■

REFERENCES

Adler, R. J., Feldman, R. E. and Taqqu, M. S., eds. (1998). *A practical guide to heavy tails. Statistical techniques and applications.* Boston: Birkhauser.

An, M. Y. (1998). Logconcavity versus logconvexity: a complete characterization. *Journal of Economic Theory* **80**, 350-369.

Andrews, D. W. K. (2003). Cross-section regression with common shocks. *Cowles Foundation Discussion Paper No. 1428*, Yale University.

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance* **9**, 203-228.

Axtell, R. L. (2001). Zipf distribution of U.S. firm sizes. *Science* **293**, 1818-1820.

Birnbaum, Z. W. (1948). On random variables with comparable peakedness. *Annals of Mathematical Statistics* **19**, 76-81.

Blattberg, R. C. and Gonedes, R. C. (1974). Comparison of stable and student distributions as statistical models for stock prices. *Journal of Business* **47**, 244-280.

Cambanis, S., Keener, R. and Simons, G. (1983). On $\alpha-$symmetric distributions. *Journal of Multivariate Analysis* **13**, 213-233.

Chan, W., Park, D. H. and Proschan, F. (1989). Peakedness of weighted averages of jointly distributed random variables. In: *Contributions to Probability and Statistics* (Gleser, L. J., Perlman, M. D., Press, S. J. and Sampson, A. R., eds.), 58-62. New York: Springer.

Duffie, D. and Pan, J. (1997). An overview of value at risk. *Journal of Derivatives* **4**, 7-49.

Eaton, M. L. (1970). A note on symmetric Bernoulli random variables. *Annals of Mathematical Statistics* **41**, 1223-1226.

Eaton, M. L. (1988). Concentration inequalities for Gauss-Markov estimators. *Journal of Multivariate Analysis* **25**, 119-138.

Efron, B. (1969). Student's t-test under symmetry conditions. *Journal of the American Statistical Association* **64**, 1278-1302.

Embrechts, P., McNeil, A., and Straumann, D. (2002). Correlation and dependence in risk management: properties and pitfalls. In: *Risk Management: Value at Risk and Beyond* (M. A. H. Dempster, ed.), Cambridge University Press, Cambridge, 176-223.

Fama, E. (1965). The behavior of stock market prices. *Journal of Business* **38**, 34-105.

Fang, K.-T., Kotz, S. and Ng, K. W. (1990). *Symmetric multivariate and related distributions.* Chapman and Hall, New York.

Fang, H. and Norman, P. (2003). To bundle or not to bundle. *Cowles Foundation Discussion Paper No. 1440*, Yale University, and *SSRI Working Paper No. 2003-18*, Department of Economics, University of Wisconsin-Madison.

Fölmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics* **6**, 429-447.

Frittelli, M. and Gianin, E. R. (2002): Putting order in risk measures. *Journal of Banking & Finance* **26**, 1473-1486.

Gabaix, X. (1999a). Zipf's law and the growth of cities. *American Economic Review* **89**, 129-132.

Gabaix X. (1999b). Zipf's law for cities: An explanation. *Quarterly Journal of Economics* **114**, 739-767.

Gabaix, X., Gopikrishnan, P., Plerou, V. and Stanley, H. E. (2003). A theory of power-law distributions in financial market fluctuations. *Nature* **423**, 267-270.

Glasserman, P., Heidelberger, P., Shahabuddin, P. (2002). Portfolio value-at-risk with heavy-tailed risk factors. *Mathematical Finance* **12**, 239–269.

Gneiting, T. (1998). On $\alpha-$symmetric multivariate characteristic functions. *Journal of Multivariate Analysis* **64**, 131-147.

Huber, P. (1981). *Robust statistics.* New York: Wiley.

Jansen, D. W. and De Vries, C. G. (1991). On the frequency of large stock returns: Putting booms and busts into perspective. *Review of Economics and Statistics* **73**,18-32.

Jensen, D. R. (1997). Peakedness of linear forms in ensembles and mixtures. *Statistics and Probability Letters* **35**, 277-282.

Jovanovic, B. and Rob, R. R. (1987). Demand-driven innovation and spatial competition over time. *Review of Economic Studies* **54**, 63-72.

Karlin, S. (1968). *Total positivity.* Stanford University Press.

Karlin, S. (1984). Mathematical models, problems, and controversies of evolutionary theory. *American Mathe-*

matical Society. Bulletin. New Series **10**, 221-274.

Karlin, S. (1992). Stochastic comparisons between means and medians for i.i.d. random variables. In: *The art of statistical science. A tribute to G. S. Watson* (K. V. Mardia, ed.), Wiley, Chichester.

Loretan, M. and Phillips, P. C. B. (1994). Testing the covariance stationarity of heavy-tailed time series. *Journal of Empirical Finance* **1**, 211-248.

Ma, C. (1998). On peakedness of distributions of convex combinations. *Journal of Statistical Planning and Inference* **70**, 51-56.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business* **36**, 394-419.

Mandelbrot, B. (1997). *Fractals and scaling in finance. Discontinuity, concentration, risk.* New York: Springer-Verlag.

Marshall, A. W. and Olkin, I. (1979). *Inequalities: theory of majorization and its applications.* New York: Academic Press.

McCulloch, J. H. (1996). Financial applications of stable distributions. In: *Handbook of statistics* (G. S. Maddala and C. R. Rao, eds.), Vol. 14 (Amsterdam: Elsevier), 393-425.

McCulloch, J. H. (1997). Measuring tail thickness to estimate the stable index alpha: A critique. *Journal of Business and Economic Statistics* **15**, 74-81.

Meerschaert, M. M. and Scheffler, H.-P. (2000). Portfolio modelling with heavy tailed random vectors. In: *Handbook of Heavy-Tailed Distributions in Finance* (S. T. Rachev, ed.), Elsevier North-Holland, New York, 595-640.

Praetz, P. (1972). The distribution of share price changes. *Journal of Business* **45**, 49-55.

Proschan, F. (1965). Peakedness of distributions of convex combinations. *Annals of Mathematical Statistics* **36**, 1703-1706.

Rachev, S. T. and Mittnik, S. (2000). *Stable Paretian Models in Finance.* New York: Wiley.

Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic orders and their applications.* Boston: Academic Press.

Tong, Y. L. (1994). Some recent developments on majorization inequalities in probability and statistics. *Linear Algebra and its Applications* **199**, 69-90.

Uchaikin, V. V. and Zolotarev, V. M. (1999). *Chance and stability. Stable distributions and their applications.* Utrecht: VSP.

Zolotarev, V. M. (1986). *One-dimensional stable distributions.* Providence: American Mathematical Society.